

FROM BOLTZMANN TO INCOMPRESSIBLE NAVIER-STOKES ON THE TORUS

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ABSTRACT. In this article we study the Boltzmann Equation, depending on the Knudsen number, in the Navier-Stokes perturbative setting on the torus in dimension N . Using the tools of the hypocoercivity theory, we prove existence and exponential decay results for the solution of this linearized equation, with explicit regularity bounds and rates. These results are uniform in the Knudsen number and then allow us to obtain a strong derivation of the incompressible Navier-Stokes equations as the Knudsen number tends to 0, in the case of Maxwellian molecules. Moreover, our method shows that the smaller the Knudsen number, the less control on the v -derivatives on the initial perturbation is needed to have existence and decay and also to deal with other kinetic models. Finally, we show that the study of this hydrodynamical limit is rather different on the torus than the already proven convergences in the whole space as it requires averaging in time, unless the initial layer conditions are satisfied.

Keywords: Boltzmann equation in the Torus, Hypocoercivity, Navier-Stokes linearization, Incompressible Navier-Stokes hydrodynamical limit, Initial layer, explicite, uniform in Knudsen number, exponential decay, rate of convergence towards global equilibrium, Kinetic Models.

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1. INTRODUCTION

This paper deals with the Boltzmann equation in a perturbative setting as the Knudsen number tends to 0. We consider the latter equation to describe the behaviour of rarefied gas particles which are moving on \mathbb{T}^N (flat torus of dimension

$N \geq 2$) with velocities in \mathbb{R}^N when the only interactions taken into account are binary collisions. More precisely, the Boltzmann equation describes the time evolution of the distribution of particles in position and velocity. A formal derivation of the Boltzmann equation from the Newton laws under the rarefied gas assumption can be found, in [6] and [7] presents Lanford Theorem (see [17]) which rigorously proves the derivation in short times.

We consider here the following more general form of it, where we denoted the Knudsen number by ε .

$$(1.1) \quad \begin{aligned} \partial_t f + v \cdot \nabla_x f &= \frac{1}{\varepsilon} Q(f, f), \text{ on } \mathbb{T}^N \times \mathbb{R}^N \\ &= \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi(|v - v_*|) b(\cos\theta) [f' f'_* - f f_*] dv_* d\sigma, \end{aligned}$$

where f' , f_* , f'_* and f are the values taken by f at v' , v_* , v'_* and v respectively. We defined:

$$\begin{cases} v' &= \frac{v+v_*}{2} + \frac{|v-v_*|}{2} \sigma \\ v'_* &= \frac{v+v_*}{2} - \frac{|v-v_*|}{2} \sigma \\ \theta &= \text{angle}(v - v_*, \sigma) \end{cases}.$$

One can find in [6], [7] or [10] that the global equilibria for the Boltzmann equation are the *Maxwellians* $M(v)$, which are gaussian density functions. Without loss of generality we can consider only the case of normalized Maxwellians:

$$M(v) = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|v|^2}{2}}.$$

The bilinear operator $Q(g, h)$ is given by

$$Q(g, h) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi(|v - v_*|) b(\cos\theta) [h' g'_* - h g_*] dv_* d\sigma.$$

1.1. The problem and its motivations. The Knudsen number is the inverse of the average number of collisions for each particle per unit of time. Therefore, as reviewed in [27], one can expect a convergence from the Boltzmann model towards the acoustics and the fluid dynamics as the Knudsen number tends to 0. The latter convergence is to be specified. However these different models describe physical phenomenon that do not evolve at the same timescale. As suggested in previous studies (see [27], [10], [24]) we can rescale in time our original equation, by a factor ε , to get rid off those time scale differences. Moreover, they also suggested that a perturbation of order ε around the global equilibrium $M(v)$ should approximate, as the Knudsen number tends to 0, the incompressible Navier-Stokes equations.

Therefore we will study the following equation

$$(1.2) \quad \partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon^2} Q(f_\varepsilon, f_\varepsilon), \text{ on } \mathbb{T}^N \times \mathbb{R}^N,$$

under the linearization $f_\varepsilon(t, x, v) = M(v) + \varepsilon M^{1/2}(v)h_\varepsilon(t, x, v)$. This leads us to the linearized Boltzmann equation, that we will study thoroughly,

$$(1.3) \quad \partial_t h_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x h_\varepsilon = \frac{1}{\varepsilon^2} L(h_\varepsilon) + \frac{1}{\varepsilon} \Gamma(h_\varepsilon, h_\varepsilon).$$

The following definitions have been used above:

$$\begin{cases} L(h) &= \left[Q(M, M^{\frac{1}{2}}h) + Q(M^{\frac{1}{2}}h, M) \right] M^{-\frac{1}{2}} \\ \Gamma(g, h) &= \frac{1}{2} \left[Q(M^{\frac{1}{2}}g, M^{\frac{1}{2}}h) + Q(M^{\frac{1}{2}}h, M^{\frac{1}{2}}g) \right] M^{-\frac{1}{2}}. \end{cases}$$

All along this paper we consider the Boltzmann equation with *hard potential*, that is to say there is a constant $C_\Phi > 0$ such that

$$\Phi(z) = C_\Phi z^\gamma, \quad \gamma \in [0, 1],$$

and a strong form of Grad's *angular cutoff* (see [11]), expressed here by the fact that we assume b to be C^1 with the controls from above

$$\forall z \in [-1, 1], \quad b(z), b(z') \leq C_b,$$

b and Φ being defined in equation (1.1).

1.2. Notations. For now on we will use the following notations, for two multi-indexes j and l in \mathbb{N}^N we define:

- $\partial_l^j = \partial_{v_j} \partial_{x_l}$,
- for i in $\{1, \dots, N\}$ we denote $c_i(j)$ the i^{th} coordinate of j ,
- the length of j will be written $|j| = \sum_i c_i(j)$,
- the multi-index δ_{i_0} by : $c_i(\delta_{i_0}) = 1$ if $i = i_0$ and 0 elsewhere.

We will write the spaces we are working on $L_{x,v}^p = L^p(\mathbb{T}^N \times \mathbb{R}^N)$, $L_x^p = L^p(\mathbb{T}^N)$ and $L_v^p = L^p(\mathbb{R}^N)$. The Sobolev's spaces $H_{x,v}^k$, H_x^k and H_v^k are defined in the same way and we denote the standard Sobolev's norm by $\|\cdot\|_{H_{x,v}^k}^2 = \sum_{|j|+|l| \leq k} \|\partial_l^j \cdot\|_{L_{x,v}^2}^2$.

1.3. Our strategy and results. The first aim of this article is to deeply study the equation (1.3) in order to obtain existence and exponential decay of solutions in Sobolev spaces $H_{x,v}^k$, independently of the Knudsen number ε . Moreover, we want all the required smallness assumptions and rates of convergence to be explicit.

Our strategy is to build a norm on standard Sobolev spaces which is equivalent to the standard norm and which satisfies a Gronwall's type inequality.

First, we construct a functional on $H_{x,v}^k$ by considering a linear combination of $\|\partial_l^j \cdot\|_{L_{x,v}^2}^2$, for all $|j|+|l| \leq k$, together with product terms of the form $\langle \partial_{l-\delta_i}^{\delta_i} \cdot, \partial_l^0 \cdot \rangle_{L_{x,v}^2}$. The distortion of the standard norm by the addition of mixed terms is necessary to have a relaxation, due to the hypocoercivity property of the linearized Boltzmann equation (1.3) (see [20]).

Then we study the flow of this functional along time for solutions to the linearized Boltzmann equation (1.3). This flow is controlled by energy estimates and, finally, a non trivial choice of coefficients in the functional yields an equivalence between the functional and the standard Sobolev norm as well as a Gronwall's type inequality, both of them being independent of ε .

We first apply this strategy to the linear case (i.e. without considering the bilinear remainder term) and prove that it generates a strong semigroup with a spectral gap and, therefore, an exponential relaxation (Theorem 2.1). Then we extend that method to the full nonlinear model and we obtain an a priori estimate on our functional (Proposition 2.2). This estimate enables us to prove the existence of solutions to the Cauchy problem and their exponential decay as long as the initial data is small enough with a smallness not depending on ε (Theorem 2.3). We can emphasize here that, thanks to the functional we used, the smaller ε the less control is needed on the v -derivatives of the initial data.

However, these results seem to tell us that the v -derivatives of solutions to equation (1.3) can blow-up as ε tends to 0. Thus, the last step is to create a new functional, based on the microscopic part of solutions, satisfying the same properties but controlling the v -derivatives as well. The fact that we ask for a control on the microscopic part of solutions to equation (1.3) is due to the deep structure of the linear operator L . This leads to the expected exponential decay independently of ε even for those v -derivatives (Theorem 2.4).

Theorem 2.3 tells us that for all ε we can build a solution h_ε to the linearized Boltzmann equation (1.3), as long as the initial perturbation is small enough independently of ε . We can then consider the sequence $(h_\varepsilon)_{0 < \varepsilon \leq 1}$ and study its limit. It appears that it converges weakly in $L_t^\infty H_x^k L_v^2$, for $k \geq k_0 > N/2$, towards a function h . Furthermore, we have the following form for h (see [3])

$$h(t, x, v) = \left[\rho(t, x) + v \cdot u(t, x) + \frac{1}{2}(|v|^2 - N)\theta(t, x) \right] M(v)^{1/2},$$

of which physical observables are solution of the linearized incompressible Navier-Stokes equations (p being the pressure function, ν and κ being constants determined by L , see Theorem 5 in [10])

$$\begin{aligned} \partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p &= 0, \\ \nabla \cdot u &= 0, \\ \partial_t \theta - \kappa \Delta \theta + u \cdot \nabla \theta &= 0, \end{aligned} \tag{1.4}$$

together with the Boussineq relation

$$\nabla(\rho + \theta) = 0. \tag{1.5}$$

However, in order to know the initial data of these quantities we study the Fourier transform on the torus of our linear operator and use it in Duhamel's formula. This gives us a strong convergence results on the time average of h_ε with an explicit rate of convergence (Theorem 2.5).

1.4. Comparison with previous results. For physical purposes, one may assume that $\varepsilon = 1$ which is a mere normalization and that is why a lot of articles about the linearized Boltzmann equation only deal with this case. The associated Cauchy problem has been worked on over the past fifty years, starting with Grad [12], and it has been studied in different spaces such as weighted $L_v^2(H_x^l)$ spaces ([25]) or weighted Sobolev spaces ([15], [13], [29]). Other results have also been proved in the

whole space instead of the torus, see for instance [1], [8] or [23], but it will not be the purpose of this article.

Our article explicitly deals with the general case for ε and prove results that are uniform in ε , allowing us to consider the hydrodynamical limit as the Knudsen number tends to 0. To solve the Cauchy problem we used an iterative scheme as in the paper mentioned above but our strategy yields a condition for the existence of solutions in $H_{x,v}^k$ (without any weight) which is uniform in ε (Theorem 6.3). In order to obtain such a result we had to consider more precise estimates on the bilinear operator Γ , depending on the existence of v -derivatives or not. Bardos and Ukai [3] obtained a similar result in the whole space but in weighted Sobolev spaces and did not prove any decay.

The study of the behaviour of such global in time solutions has also been studied. Guo worked in weighted Sobolev's spaces and proved the boundedness of solutions to equation (1.3) in [15] as well as an exponential decay (uniform in ε) in [16]. The norm involved in [15, 16] is quite intricate and requires a lot of technical computations. To avoid specific and technical calculations, the theory of hypocoercivity (see [19]) focuses on the properties of the Boltzmann operator and which are quite similar to hypoellipticity. This theory has been used in [20] to obtain exponential decay in standard Sobolev spaces in the case $\varepsilon = 1$.

We used the idea of Mouhot and Neumann developed in [20] consisting in considering a functional on $H_{x,v}^k$ involving mixed scalar products. Thus, in this paper we construct such a quadratic form but with coefficient depending on ε . Working in the general case for ε yields new calculations and we had to consider some orthogonal properties of the bilinear operator Γ to overcome those issues. Moreover we had to construct a new norm out of this functional which controls the v -derivatives by a factor ε .

Then, the fact that the study yields a norm containing some ε factors prevented us from having a uniform exponential decay for the v -derivatives. We used the idea of Guo, in [16], of looking at the microscopic part of the solution h_ε everytime we look at a differentiation in v . This idea catches the interesting structure of L on its orthogonal part. Combining this idea with our previous strategy fills the gap for the v -derivatives.

Finally, our uniform results enables us to derive a weak convergence in $H_x^k L_v^2$ towards solutions to the incompressible Navier-Stokes equations, together with the Boussineq relation. We then find a way to obtain strong convergence using the ideas of the Fourier study of the linear operator $L - v \cdot \nabla_x$, developed in [3] and [9], combined with a Duhamel formula. However, the study done in [3] strongly relies on an argument of stationary phase developed in [26] which is no longer applicable in the torus. Indeed, the Fourier space in the whole space is continuous and so integration by parts can be done in that frequency space. This tool is no longer available in the frequency space of the torus which is discrete.

Our Theorem 2.5 shows that the behaviour of the hydrodynamical limit is quite different on the torus where an averaging in time is necessary for general initial data. However, we obtain the same relation between the limit at $t = 0$ and the initial perturbation h_{in} and also the existence of an initial layer. That is to say that we can have a convergence in $L_{[0,T]}^2 = L^2([0, T])$ if and only if the initial perturbation

satisfies some physical properties, which appear to be the same as in the whole space \mathbb{R}^N studied in [3].

Finally, results that do not involve hydrodynamical limit (existence and exponential decay results) are applicable to a bigger class of operators. In Appendix 1 we prove that those theorems also holds for other kinetic collisional models such as the linear relaxation, the semi-classical relaxation, the linear Fokker-Planck equation and the Landau equation with hard and moderately soft potential.

1.5. Organization of the paper. The next section 2 is divided in two different subsections.

As announced above, we are going to use the hypocoercivity of the Boltzmann equation (1.1). This hypocoercivity can be described in terms of technical properties on L and Γ and, in order to obtain more general results, we consider them as a basis of our paper. Thus, subsection 2.1 gives them in details and a proof of the fact that L and Γ indeed satisfy those properties is given in Appendix 1. Most of them as been proved and used in [20] but we required more precise ones to deal with the general case.

The second subsection 2.2 is dedicated to a mathematical formulation of the results described in subsection 1.3.

As said when we described our strategy (subsection 1.3), we are going to study the flow of a functional involving $L_{x,v}^2$ -norm of x and v derivatives and mixed scalar products. To control this flow in time we compute energy estimates for each of these terms in a toolbox (section 3) which will be used and referred to all along the rest of the paper. Proofs of those energy estimates are given in Appendix 2.

Finally, sections 4, 5, 6, 7 and 8 are the proofs respectively of Theorem 2.1 (about the strong semigroup property of the linear part of equation (1.3)), Proposition 2.2 (an a priori estimates on the constructed functional for the full model), Theorem 2.3 (existence and exponential decay of solutions to equation (1.3)), Theorem 2.4 (showing the uniform boundedness of the v -derivatives) and of Theorem 2.5 (dealing with the hydrodynamical limit).

We notice here that section 6 is divided in two subsection. Subsection 6.1 deals with the existence of solutions for all $\varepsilon > 0$ and subsection 6.2 proved the exponential decay of those solutions.

2. MAIN RESULTS

This section is divided in two parts. The first one translate the hypocoercivity aspects of the Boltzmann operator in terms of mathematical properties for L and Γ . Then, the second one states our results in terms of those assumptions.

2.1. Hypocoercivity assumptions. This section is dedicated to the framework and assumptions of the hypocoercivity theory. A state of the art of this theory can be found in [19].

2.1.1. Assumptions on the linear operator L .

Assumptions in $H_{x,v}^1$:

(H1): Coercivity and general controls

$L : L_v^2 \longrightarrow L_v^2$ is a closed and self-adjoint operator with $L = K - \Lambda$ such that:

- Λ is coercive:

- it exists $\|\cdot\|_{\Lambda_v}$ norm on L_v^2 such that

$$\forall h \in L_v^2, \nu_0^\Lambda \|h\|_{L_v^2}^2 \leq \nu_1^\Lambda \|h\|_{\Lambda_v}^2 \leq \langle \Lambda(h), h \rangle_{L_v^2} \leq \nu_2^\Lambda \|h\|_{\Lambda_v}^2,$$

- Λ has a defect of coercivity regarding its v derivatives:

$$\forall h \in H_v^1, \langle \nabla_v \Lambda(h), \nabla_v h \rangle_{L_v^2} \geq \nu_3^\Lambda \|\nabla_v h\|_{\Lambda_v}^2 - \nu_4^\Lambda \|h\|_{\Lambda_v}^2.$$

- There exists $C^L > 0$ such that

$$\forall h \in L_v^2, \forall g \in L_v^2, \langle L(h), g \rangle_{L_v^2} \leq C^L \|h\|_{\Lambda_v} \|g\|_{\Lambda_v},$$

where $(\nu_k^\Lambda)_{1 \leq k \leq 4}$ are strictly positive constants depending on the operator and the dimension of the velocities space N .

As in [20], we define a new norm on $L_{x,v}^2$:

$$\|\cdot\|_\Lambda = \|\|\cdot\|_{\Lambda_v}\|_{L_x^2}.$$

(H2): Mixing property in velocity

$$\forall \delta > 0, \exists C(\delta) > 0, \forall h \in H_v^1, \langle \nabla_v K(h), \nabla_v h \rangle_{L_v^2} \leq C(\delta) \|h\|_{L_v^2}^2 + \delta \|\nabla_v h\|_{L_v^2}^2.$$

(H3): Relaxation to equilibrium

We suppose that the Kernel of L is generated by d functions which form an orthonormal basis for $\text{Ker}(L)$:

$$\text{Ker}(L) = \text{Span}\{\phi_1(v), \dots, \phi_d(v)\}.$$

Moreover, we assume that the ϕ_i are of the form $P_i(v)e^{-|v|^2/4}$, where P_i is a polynomial.

Furthermore, denoting by π_L the orthogonal projector in L_v^2 on $\text{Ker}(L)$ we assume that we have the following local coercivity property:

$$\exists \lambda > 0, \forall h \in L_v^2, \langle L(h), h \rangle_{L_v^2} \leq -\lambda \|h^\perp\|_{\Lambda_v}^2,$$

where $h^\perp = h - \pi_L(h)$ denotes the microscopic part of h (the orthogonal to $\text{Ker}(L)$ in L_v^2).

We are using the same hypothesis as in [20], except that we require the ϕ_i to be of a specific form. This additional requirement allows us to derive properties on the v -derivatives of π_L that we will state in the toolbox section 3.

Then we have two more properties on L in order to deal with higher order Sobolev spaces.

Assumptions in $H_{x,v}^k$, $k > 1$:

(H1') : Default of coercivity for higher derivatives

We assume that L satisfies (H1).

For all $k \geq 1$, for all multi-indexes in \mathbb{N}^N j and l such that $k = |j| + |l|$, we define $\partial_l^j = \partial_v^j \partial_x^l$. With such notations we assume that L satisfies the following property:

$$\forall k \geq 1, \forall |j| + |l| = k, \forall h \in H_{x,v}^k, \quad \langle \partial_l^j \Lambda(h), \partial_l^j h \rangle_{L_{x,v}^2} \geq \nu_5^\Lambda \|\partial_l^j h\|_\Lambda^2 - \nu_6^\Lambda \|h\|_{H_{x,v}^{k-1}},$$

where ν_5^Λ and ν_6^Λ are strictly positive constants depending on L and N .

We also define a new norm on $H_{x,v}^k$:

$$\|\cdot\|_{H_\Lambda^k} = \left(\sum_{|j|+|l| \leq k} \|\partial_l^j \cdot\|_\Lambda^2 \right)^{1/2}$$

(H2') : Mixing properties

As above, Mouhot and Neumann extended the hypothesis (H2) to higher Sobolev's spaces:

$$\forall \delta > 0, \exists C(\delta) > 0, \forall h \in H_{x,v}^k, \quad \langle \partial_l^j K(h), \partial_l^j h \rangle_{L_{x,v}^2} \leq C(\delta) \|h\|_{H_{x,v}^{k-1}}^2 + \delta \|\partial_l^j h\|_{L_{x,v}^2}^2.$$

2.1.2. *Assumptions on the second order term Γ .* To solve our problem uniformly in ε we had to precise the hypothesis made in [20] in order to have a deeper understanding of the operator Γ . This lead us to two different assumptions.

(H4) : Control on the second order operator

$\Gamma : L_v^2 \times L_v^2 \longrightarrow L_v^2$ is a bilinear symmetric operator such that for all multi-indexes j and l such that $|j| + |l| \leq k$, $k \geq 0$,

$$\left| \langle \partial_l^j \Gamma(g, h), f \rangle_{L_{x,v}^2} \right| \leq \begin{cases} \mathcal{G}_{x,v}^k(g, h) \|f\|_\Lambda & , \text{ if } j \neq 0 \\ \mathcal{G}_x^k(g, h) \|f\|_\Lambda & , \text{ if } j = 0 \end{cases},$$

$\mathcal{G}_{x,v}^k$ and \mathcal{G}_x^k being such that $\mathcal{G}_{x,v}^k \leq \mathcal{G}_{x,v}^{k+1}$, $\mathcal{G}_x^k \leq \mathcal{G}_x^{k+1}$ and satisfying the following property:

$$\exists k_0 \in \mathbb{N}, \forall k \geq k_0, \exists C_\Gamma > 0, \quad \begin{cases} \mathcal{G}_{x,v}^k(g, h) \leq C_\Gamma \left(\|g\|_{H_{x,v}^k} \|h\|_{H_\Lambda^k} + \|h\|_{H_{x,v}^k} \|g\|_{H_\Lambda^k} \right) \\ \mathcal{G}_x^k(g, h) \leq C_\Gamma \left(\|h\|_{H_x^k L_v^2} \|g\|_{H_\Lambda^k} + \|g\|_{H_x^k L_v^2} \|h\|_{H_\Lambda^k} \right). \end{cases}$$

(H5) : Orthogonality to the Kernel of the linear operator

$$\forall h, g \in \text{Dom}(\Gamma) \cap L_v^2, \quad \Gamma(g, h) \in \text{Ker}(L)^\perp.$$

2.2. Statement of the Theorems.

2.2.1. *Uniform result for the linear Boltzmann equation.* For k in \mathbb{N}^* and some constants $(b_{j,l}^{(k)})_{j,l}$, $(\alpha_l^{(k)})_l$ and $(a_{i,l}^{(k)})_{i,l}$ strictly positive and $0 < \varepsilon \leq 1$ we define the following functional on $H_{x,v}^k$, where we emphasize that there is a dependance on ε ,

which is the key point of our study.

$$\|\cdot\|_{\mathcal{H}_\varepsilon^k} = \left[\sum_{\substack{|j|+|l|\leq k \\ |j|\geq 1}} b_{j,l}^{(k)} \varepsilon^2 \|\partial_l^j \cdot\|_{L_{x,v}^2}^2 + \sum_{|l|\leq k} \alpha_l^{(k)} \|\partial_l^0 \cdot\|_{L_{x,v}^2}^2 + \sum_{\substack{|l|\leq k \\ i, c_i(l)>0}} a_{i,l}^{(k)} \varepsilon \langle \partial_{l-\delta_i}^{\delta_i} \cdot, \partial_l^0 \cdot \rangle_{L_{x,v}^2} \right]^{\frac{1}{2}}.$$

We first studied the linearized equation (1.3), without taking into account the bilinear remainder operator. By letting π_w be the projector in $L_{x,v}^2$ onto $\text{Ker}(w)$ we obtained the following semigroup property for L .

Theorem 2.1. *If L is a linear operator satisfying the conditions (H1'), (H2') and (H3) then it exists $0 < \varepsilon_N \leq 1$ such that for all k in \mathbb{N}^* ,*

- (1) *for all $0 < \varepsilon \leq \varepsilon_N$, $G_\varepsilon = \varepsilon^{-2}L - \varepsilon^{-1}v \cdot \nabla_x$ generates a C^0 -semigroup on $H_{x,v}^k$.*
- (2) *there exist $C_G^{(k)}$, $(b_{j,l}^{(k)})$, $(\alpha_l^{(k)})$, $(a_{i,l}^{(k)}) > 0$ such that for all $0 < \varepsilon \leq \varepsilon_N$:*

$$\|\cdot\|_{\mathcal{H}_\varepsilon^k}^2 \sim \left(\|\cdot\|_{L_{x,v}^2}^2 + \sum_{|l|\leq k} \|\partial_l^0 \cdot\|_{L_{x,v}^2}^2 + \varepsilon^2 \sum_{\substack{|l|+|j|\leq k \\ |j|\geq 1}} \|\partial_l^j \cdot\|_{L_{x,v}^2}^2 \right),$$

and for all h in $H_{x,v}^k$,

$$\langle G_\varepsilon(h), h \rangle_{\mathcal{H}_\varepsilon^k} \leq -C_G^{(k)} \|h - \pi_{G_\varepsilon}(h)\|_{H_\Lambda^k}^2.$$

This theorem gives us an exponential decay for the semigroup generated by G_ε .

2.2.2. Uniform perturbative result for the Boltzmann equation. The next result states that if we add the bilinear remainder operator then it is enough, if ε is small enough, to slightly change our new norm to have a control on the solution.

Proposition 2.2. *If L is a linear operator satisfying the conditions (H1'), (H2') and (H3) and Γ a bilinear operator satisfying (H4) and (H5) then it exists $0 < \varepsilon_N \leq 1$ such that for all k in \mathbb{N}^* ,*

- (1) *there exist $K_0^{(k)}$, $K_1^{(k)}$, $K_2^{(k)}$ $(b_{j,l}^{(k)})$, $(\alpha_l^{(k)})$, $(a_{i,l}^{(k)}) > 0$, independent of Γ and ε , such that for all $0 < \varepsilon \leq \varepsilon_N$:*

$$\|\cdot\|_{\mathcal{H}_\varepsilon^k}^2 \sim \left(\|\cdot\|_{L_{x,v}^2}^2 + \sum_{|l|\leq k} \|\partial_l^0 \cdot\|_{L_{x,v}^2}^2 + \varepsilon^2 \sum_{\substack{|l|+|j|\leq k \\ |j|\geq 1}} \|\partial_l^j \cdot\|_{L_{x,v}^2}^2 \right),$$

- (2) *and for all h_{in} in $H_{x,v}^k \cap \text{Ker}(G_\varepsilon)^\perp$ and all g in $\text{Dom}(\Gamma) \cap H_{x,v}^k$, if we have a solution h in $H_{x,v}^k$ to the following equation*

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon} \Gamma(g, h),$$

then

$$\frac{d}{dt} \|h\|_{\mathcal{H}_\varepsilon^k}^2 \leq -K_0^{(k)} \|h\|_{H_\Lambda^k}^2 + K_1^{(k)} (\mathcal{G}_x^k(g, h))^2 + \varepsilon^2 K_2^{(k)} (\mathcal{G}_{x,v}^k(g, h))^2.$$

One can remark that the norm constructed above leaves free the x -derivatives while it controls the v ones by a factor ε .

We want to emphasize here that this result shows that the derivative of the norm is control by the x -derivatives of Γ and the Sobolev norm of Γ , but weakened by a factor ε^2 . This is important as our norm $\|\cdot\|_{\mathcal{H}_\varepsilon^k}^2$ controls the $L_v^2(H_x^k)$ -norm by a factor of order 1 whereas it controls the whole $H_{x,v}^k$ -norm by a multiplicative factor of order $1/\varepsilon$. These thoughts lead to a new approximation of the solution, in which we consider the microscopic projection of a new operator $\tilde{\Gamma}$, to the linearized Boltzmann equation and then to our perturbative result.

Theorem 2.3. *Let Q be a bilinear operator such that:*

- *the equation (1.2) admits an equilibrium $0 \leq M \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$,*
- *the linearized operator $L = L(h)$ around M with the scaling $f = M + \varepsilon M^{1/2}h$ satisfies (H1'), (H2') and (H3),*
- *the bilinear remaining term $\Gamma = \Gamma(h, h)$ in the linearization satisfies (H4) and (H5).*

Then there exists $0 < \varepsilon_N \leq 1$ such that for any $k \geq k_0$ (defined in (H4)),

- (1) *there exist $(b_{j,l}^{(k)}), (\alpha_l^{(k)}), (a_{i,l}^{(k)}) > 0$, independent of Γ and ε , such that for all $0 < \varepsilon \leq \varepsilon_N$:*

$$\|\cdot\|_{\mathcal{H}_\varepsilon^k}^2 \sim \left(\|\cdot\|_{L_{x,v}^2}^2 + \sum_{|l| \leq k} \|\partial_l^0 \cdot\|_{L_{x,v}^2}^2 + \varepsilon^2 \sum_{\substack{|l|+|j| \leq k \\ |j| \geq 1}} \|\partial_l^j \cdot\|_{L_{x,v}^2}^2 \right),$$

- (2) *there exist $\delta_k > 0$, $C_k > 0$ and $\tau_k > 0$ such that for all $0 < \varepsilon \leq \varepsilon_N$:*

For any distribution $0 \leq f_{in} \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$ with $f_{in} = M + \varepsilon M^{1/2}h_{in}$, h_{in} in $\text{Ker}(G_\varepsilon)^\perp$ and

$$\|h_{in}\|_{\mathcal{H}_\varepsilon^k} \leq \delta_k,$$

there exists a unique global smooth (in $H_{x,v}^k$, continuous in time) solution $0 \leq f_\varepsilon = f_\varepsilon(t, x, v)$ to equation (1.2) which, moreover, satisfies $f_\varepsilon = f_\infty + \varepsilon f_\infty^{1/2}h_\varepsilon$ with:

$$\|h_\varepsilon\|_{\mathcal{H}_\varepsilon^k} \leq \|h_{in}\|_{\mathcal{H}_\varepsilon^k} e^{-\tau_k t}.$$

The fact that we are asking h_{in} to be in $\text{Ker}(G_\varepsilon)^\perp$ just states that we want f_{in} to have the same physical quantities as the global equilibrium M . This is a compulsory requirement as one can easily check that the physical quantities

$$\int_{\mathbb{T}^N \times \mathbb{R}^N} f_\varepsilon(x, v) dx dv, \quad \int_{\mathbb{T}^N \times \mathbb{R}^N} v f_\varepsilon(x, v) dx dv, \quad \int_{\mathbb{T}^N \times \mathbb{R}^N} |v|^2 f_\varepsilon(x, v) dx dv$$

are preserved with time (see [7] for instance).

Notice that the $\mathcal{H}_\varepsilon^k$ -norm in this theorem is the same than the one we constructed in Proposition 2.2

2.2.3. The boundedness of the v -derivatives. As a corollary we have that the $H_x^k(L_v^2)$ -norm decays exponentially independently of ε but that the only control we have on the $H_{x,v}^k$ is

$$\|h_\varepsilon\|_{H_{x,v}^k} \leq \frac{\delta_k}{\varepsilon} e^{-\tau_k t}.$$

This seems to tell us that the v -derivatives can blow-up at a rate $1/\varepsilon$. However, without proving the existence in the norm he used, Guo, in [16], showed that one can prove that there is no explosion. By controlling only the microscopic part we reached the same conclusion.

We define the following positive quadratic form

$$\|\cdot\|_{\mathcal{H}_{\varepsilon\perp}^k}^2 = \sum_{\substack{|j|+|l|\leq k \\ |j|\geq 1}} b_{j,l}^{(k)} \|\partial_l^j (Id - \pi_L)\|_{L_{x,v}^2}^2 + \sum_{|l|\leq k} \alpha_l^{(k)} \|\partial_l^0 \cdot\|_{L_{x,v}^2}^2 + \sum_{\substack{|l|\leq k \\ i, c_i(l)>0}} a_{i,l}^{(k)} \varepsilon \langle \partial_{l-\delta_i}^{\delta_i} \cdot, \partial_l^0 \cdot \rangle_{L_{x,v}^2}.$$

Theorem 2.4. *Under the same conditions as in Theorem 2.3, for all $k \geq k_0$, there exist $(b_{j,l}^{(k)})$, $(\alpha_l^{(k)})$, $(a_{i,l}^{(k)}) > 0$ and $0 < \varepsilon_N \leq 1$ such that for all $0 < \varepsilon \leq \varepsilon_N$:*

- (1) $\|\cdot\|_{\mathcal{H}_{\varepsilon\perp}^k} \sim \|\cdot\|_{H_{x,v}^k}$, independently of ε ,
- (2) if h_ε is a solution of 1.3 in $H_{x,v}^k$ with $\|h_{in}\|_{\mathcal{H}_{\varepsilon\perp}^k} \leq \delta'_k$ then

$$\|h_\varepsilon\|_{\mathcal{H}_{\varepsilon\perp}^k} \leq \delta'_k e^{-\tau'_k t},$$

where δ'_k and τ'_k are strictly positive constants independent of ε .

Our theorem states that one can really expect a convergence of solutions of collisional kinetic models near equilibrium towards a solution of fluid dynamics equations. Indeed, the smallness assumption on the initial perturbation does not depend on the parameter ε as long as ε is small enough.

2.2.4. The hydrodynamical limit on the torus for Maxwellian particles. We then define the following macroscopic quantities

- the particles density $\rho_\varepsilon(t, x) = \langle M(v)^{1/2}, h_\varepsilon(t, x, v) \rangle_{L_v^2}$,
- the mean velocity $u_\varepsilon(t, x) = \langle v M(v)^{1/2}, h_\varepsilon(t, x, v) \rangle_{L_v^2}$,
- the temperature $\theta_\varepsilon(t, x) = \frac{1}{N} \langle (|v|^2 - N) M(v)^{1/2}, h_\varepsilon(t, x, v) \rangle_{L_v^2}$.

The theorem 2.3 tell us that, for $k \geq k_0$, the sequence $(h_\varepsilon)_{\varepsilon>0}$ converges (up to an extraction) weakly-* in $L_t^\infty(H_l^k L_v^2)$ towards a function h . Such a weak convergence enables us to use the theorem 1.1 of [3] to get that

- (1) h is in $\text{Ker}(L)$, so of the form

$$h(t, x, v) = \left[\rho(t, x) + v \cdot u(t, x) + \frac{1}{2}(|v|^2 - N)\theta(t, x) \right] M(v)^{1/2},$$

- (2) $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ converges weakly* in $L_t^\infty(H_x^k)$ towards (ρ, u, θ) ,
- (3) (ρ, u, θ) satisfies the incompressible Navier-Stokes equations (1.4) as well as the Boussineq equation (1.5).

If such a result confirms the fact that one can derive the incompressible Navier-Stokes equations from the Boltzmann equation, it does unfortunately neither give us the continuity of h nor the initial condition verified by (ρ, u, θ) , depending on $(\rho_{in}, u_{in}, \theta_{in})$, macroscopic quantities associated to h_{in} . Our next, and final step, is therefore to link the last two triplets and so to understand the convergence $h_\varepsilon \rightarrow h$ more deeply. This is the purpose of the next, and last, theorem.

In Remark 8.10, we define $V_T(\varepsilon)$ and prove the following result

$$\forall T > 0, \quad V_T(\varepsilon) = \sup_{t \in [0, T]} \|h_\varepsilon - h\|_{L_x^\infty L_v^2} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

Theorem 2.5. Consider $k \geq k_0$ and h_{in} in $H_{x,v}^k$ such that $\|h_{in}\|_{\mathcal{H}_\varepsilon^k} \leq \delta_k$.

Then, $(h_\varepsilon)_{\varepsilon>0}$ exists for all $0 < \varepsilon \leq \varepsilon_N$ and converges weakly* in $L_t^\infty(H_x^k L_v^2)$ towards h such that $h \in \text{Ker}(L)$, with $\nabla_x \cdot u = 0$ and $\rho + \theta = 0$.

Furthermore, $\int_0^T h dt$ belongs to $H_x^k L_v^2$ and it exists $C > 0$ such that for all $T > 0$,

$$\left\| \int_0^T h dt - \int_0^T h_\varepsilon dt \right\|_{H_x^k L_v^2} \leq C \max\{\sqrt{\varepsilon}, T^{3/2} \sqrt{\varepsilon}, TV_T(\varepsilon)\}.$$

One can have a strong convergence in $L_{[0,T]}^2 H_x^k L_v^2$ only if h_{in} is in $\text{Ker}(L)$ with $\nabla_x \cdot u_{in} = 0$ and $\rho_{in} + \theta_{in} = 0$.

This theorem gives us strong convergences for $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ towards (ρ, u, θ) but above all it gives us that (ρ, u, θ) is the solution to the incompressible Navier-Stokes equations together with the Boussineq equation satisfying the initial conditions:

- $u(0, x) = Pu_{in}(x)$, where $Pu_{in}(x)$ is the divergence-free part of $u_{in}(x)$,
- $\rho(0, x) = -\theta(0, x) = \frac{1}{2}(\rho_{in}(x) - \theta_{in}(x))$.

Finally, we mention here that the obligation of an integration in time for non special initial condition is only due to the linear part $\varepsilon^{-2}L - \varepsilon^{-1}v \cdot \nabla_x$, whereas the case $T = +\infty$ is prevented by the second order term Γ .

3. TOOLBOX: FLUID PROJECTION AND A PRIORI ENERGY ESTIMATES

In this section we are going to give some inequalities we are going to use and to refer to throughout the sequel. First we start with some properties concerning the projection in L_v^2 onto $\text{Ker}(L)$: π_L . Then, because we want to estimate all the terms appearing in the $H_{x,v}^k$ -norm to estimate the operators $\mathcal{H}_\varepsilon^k$ and $\mathcal{H}_{\varepsilon\perp}^k$, we will give upper bound on their time derivatives. The proofs are only technical and the interested reader will find them in Appendix 2.

We are assuming there that L is having properties (H1'), (H2') and (H3), that Γ satisfies (H4) and (H5) and that $0 < \varepsilon \leq 1$.

3.1. Properties concerning the fluid projection π_L . We already know that L is acting on L_v^2 , with $\text{Ker}(L) = \text{Span}(\phi_1, \dots, \phi_d)$, with $(\phi_i)_{1 \leq i \leq d}$ an orthonormal family, we obtain directly a useful formula for the orthogonal projection on $\text{Ker}(L)$ in L_v^2 , π_L :

$$(3.1) \quad \forall h \in L_v^2, \quad \pi_L(h) = \sum_{i=1}^d \left(\int_{\mathbb{R}^N} h \phi_i dv \right) \phi_i.$$

Plus, (H3) states that $\phi_i = P_i(v)e^{-|v|^2/4}$, where P_i is a polynomial. Therefore, direct computations and Cauchy-Schwarz inequality raise that π_L is continuous on $H_{x,v}^k$ with

$$(3.2) \quad \forall k \in \mathbb{N}, \exists C_{\pi k} > 0, \forall h \in H_{x,v}^k, \quad \|\pi_L(h)\|_{H_{x,v}^k}^2 \leq C_{\pi k} \|h\|_{H_{x,v}^k}^2.$$

More precisely one can find that for all k in \mathbb{N}

$$(3.3) \quad \forall |j| + |l| = k, \forall h \in H_{x,v}^k, \quad \|\partial_l^j \pi_L(h)\|_{L_{x,v}^2}^2 \leq C_{\pi k} \|\partial_l^0 \pi_L(h)\|_{H_{x,v}^k}^2.$$

Finally, building the Λ -norm one can find that in all the collisional kinetic equations concerned here we have that

$$(3.4) \quad \exists C_\pi > 0, \forall h \in L_{x,v}^2, \quad \|\pi_L(h)\|_\Lambda^2 \leq C_\pi \|h\|_{L_{x,v}^2}^2.$$

Then we can also use the properties of the torus to obtain Poincaré type inequalities. This can be very useful thanks to the next proposition, which is proved in Appendix 2.

Proposition 3.1. *Let a and b be in \mathbb{R}^* and consider the operator $G = aL - bv \cdot \nabla_x$ acting on $H_{x,v}^1$.*

If L satisfies (H1) and (H3) then

$$\text{Ker}(G) = \text{Ker}(L).$$

Therefore, if we define, for $0 < \varepsilon \leq 1$:

$$G_\varepsilon = \frac{1}{\varepsilon^2} L - \frac{1}{\varepsilon} v \cdot \nabla_x,$$

then we have a nice description of π_{G_ε} :

$$\forall h \in L_{x,v}^2, \quad \pi_{G_\varepsilon}(h) = \sum_{i=1}^d \left(\int_{\mathbb{T}^N} \int_{\mathbb{R}^N} h \phi_i dv \right) \phi_i.$$

That means that $\pi_{G_\varepsilon}(h)$ is, up to a multiplicative constant, the mean of $\pi_L(h)$ over the torus. We deduce that if h belongs to $\text{Ker}(G_\varepsilon)^\perp$, $\pi_L(h)$ has zero mean on the torus and is an operator not depending on the x variable. Thus we can apply Poincaré inequality on the torus:

$$(3.5) \quad \forall h \in \text{Ker}(G_\varepsilon)^\perp, \quad \|\pi_L(h)\|_{L_{x,v}^2}^2 \leq C_p \|\nabla_x \pi_L(h)\|_{L_{x,v}^2}^2 \leq C_p \|\nabla_x h\|_{L_{x,v}^2}^2.$$

3.2. A priori energy estimates. Our work in this article is to study the evolution of the norms involved in the definition of the operators $\mathcal{H}_\varepsilon^k$ and $\mathcal{H}_{\varepsilon\perp}^k$ and to combine them to obtain the results stated above. The Appendix 2 contains the proofs, which are technical computations together with some choices of decomposition, of the following a priori estimates. Note that all the constants K_1 , K_{dx} and K_{k-1} used in the inequalities below are independent of ε , Γ and g , and only depend constructively on the constants defined in the hypocoercivity assumptions or in the subsection above. The number e can be any positive real number and will be chosen later.

We would like to study both linear and non-linear models but they appeared to be very similar. In order to avoid long and similar inequalities we will write in parenthesis terms we need to add for the full model.

Let g be a function in $H_{x,v}^k$. We now consider a function h in $\text{Ker}(G_\varepsilon)^\perp \cap H_{x,v}^k$, for some k in \mathbb{N}^* , which is solution of the linear (linearized) Boltzmann equation:

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) \left(+ \frac{1}{\varepsilon} \Gamma(g, h) \right).$$

We remind the reader that the following notation is used: $h^\perp = h - \pi_L(h)$.

3.2.1. *Time evolutions for quantities in $H_{x,v}^1$.* We write the $L_{x,v}^2$ -norm estimate

$$(3.6) \quad \frac{d}{dt} \|h\|_{L_{x,v}^2}^2 \leq -\frac{\lambda}{\varepsilon^2} \|h^\perp\|_\Lambda^2 \left(+ \frac{1}{\lambda} (\mathcal{G}_x^0(g, h))^2 \right).$$

Then the time evolution of the x -derivatives

$$(3.7) \quad \frac{d}{dt} \|\nabla_x h\|_{L_{x,v}^2}^2 \leq -\frac{\lambda}{\varepsilon^2} \|\nabla_x h^\perp\|_\Lambda^2 \left(+ \frac{1}{\lambda} (\mathcal{G}_x^1(g, h))^2 \right),$$

and of the v -derivatives

$$(3.8) \quad \begin{aligned} \frac{d}{dt} \|\nabla_v h\|_{L_{x,v}^2}^2 &\leq \frac{K_1}{\varepsilon^2} \|h^\perp\|_\Lambda^2 + \frac{K_{dx}}{\varepsilon^2} \|\nabla_x h\|_{L_{x,v}^2}^2 - \frac{\nu_3^\Lambda}{\varepsilon^2} \|\nabla_v h\|_\Lambda^2 \\ &\quad \left(+ \frac{3}{\nu_3^\Lambda} (\mathcal{G}_{x,v}^1(g, h))^2 \right). \end{aligned}$$

Finally, we will need a control on the scalar product as well, as explained in the strategy subsection 1.3. Notice that we have some freedom as e can be any positive number.

$$(3.9) \quad \begin{aligned} \frac{d}{dt} \langle \nabla_x h, \nabla_v h \rangle_{L_{x,v}^2} &\leq \frac{C^L e}{\varepsilon^3} \|\nabla_x h^\perp\|_\Lambda^2 - \frac{1}{\varepsilon} \|\nabla_x h\|_{L_{x,v}^2}^2 + \frac{2C^L}{e\varepsilon} \|\nabla_v h\|_\Lambda^2 \\ &\quad \left(+ \frac{e}{C^L \varepsilon} (\mathcal{G}_x^1(g, h))^2 \right). \end{aligned}$$

3.2.2. *Time evolutions for quantities in $H_{x,v}^k$.* We consider multi-indexes j and l such that $|j| + |l| = k$.

As in the previous case, we have a control on the time evolution of the pure x -derivatives,

$$(3.10) \quad \frac{d}{dt} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \leq -\frac{\lambda}{\varepsilon^2} \|\partial_l^0 h^\perp\|_\Lambda^2 \left(+ \frac{1}{\lambda} (\mathcal{G}_x^k(g, h))^2 \right).$$

In the case where $|j| \geq 1$, that is to say when we have at least one derivative in v , we obtained the following upper bound

$$(3.11) \quad \begin{aligned} \frac{d}{dt} \|\partial_l^j h\|_{L_{x,v}^2}^2 &\leq -\frac{\nu_5^\Lambda}{\varepsilon^2} \|\partial_l^j h\|_\Lambda^2 + \frac{3(\nu_1^\Lambda)^2 N}{\nu_5^\Lambda (\nu_0^\Lambda)^2} \sum_{i, c_i(j) > 0} \left\| \partial_{l+\delta_i}^{j-\delta_i} h \right\|_\Lambda^2 + \frac{K_{k-1}}{\varepsilon^2} \|h\|_{H_{x,v}^{k-1}}^2 \\ &\quad \left(+ \frac{3}{\nu_5^\Lambda} (\mathcal{G}_{x,v}^k(g, h))^2 \right). \end{aligned}$$

We may find useful to consider the particular case where $|j| = 1$,

$$(3.12) \quad \frac{d}{dt} \|\partial_{l-\delta_i}^{\delta_i} h\|_{L_{x,v}^2}^2 \leq -\frac{\nu_5^\Lambda}{\varepsilon^2} \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2 + \frac{3\nu_1^\Lambda}{\nu_5^\Lambda \nu_0^\Lambda} \|\partial_l^0 h\|_{L_{x,v}^2}^2 + \frac{K_{k-1}}{\varepsilon^2} \|h\|_{H_{x,v}^{k-1}}^2 + \frac{3}{\nu_5^\Lambda} (\mathcal{G}_{x,v}^k(g, h))^2.$$

Finally we will need the time evolution of the following scalar product:

$$(3.13) \quad \frac{d}{dt} \langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2} \leq \frac{C^L e}{\varepsilon^3} \|\partial_l^0 h^\perp\|_\Lambda^2 - \frac{1}{\varepsilon} \|\partial_l^0 h\|_{L_{x,v}^2}^2 + \frac{2C^L}{e\varepsilon} \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2 + \frac{e}{C^L \varepsilon} (\mathcal{G}_x^k(g, h))^2,$$

where we still have some freedom as e is any positive number.

We just emphasize here that one can see that we were careful about which derivatives are involved in the terms that contain Γ . This is because our operator $\|\cdot\|_{\mathcal{H}_\varepsilon^k}$ controls the $H_x^k(L_v^2)$ -norm by a mere constant whereas it controls the entire $H_{x,v}^k$ -norm by a factor $1/\varepsilon$.

3.2.3. Time evolutions for orthogonal quantities in $H_{x,v}^k$. For the theorem 2.4 we are going to need four others inequalities which are a little bit more intricate as they need to know the shape of π_L as described in the subsection above. The proofs are written in Appendix 2 and we are just looking at the whole equation in the setting $g = h$.

We want the time evolution of the v -derivatives of the orthogonal (microscopic) part of h , as suggested in [16] this allows us to really take advantage of the structure of the linear operator L on its orthogonal:

$$(3.14) \quad \frac{d}{dt} \|\nabla_v h^\perp\|_{L_{x,v}^2}^2 \leq \frac{K_1^\perp}{\varepsilon^2} \|h^\perp\|_\Lambda^2 + K_{dx}^\perp \|\nabla_x h\|_{L_{x,v}^2}^2 - \frac{\nu_3^\Lambda}{2\varepsilon^2} \|\nabla_v h^\perp\|_\Lambda^2 + \frac{3}{\nu_3^\Lambda} (\mathcal{G}_{x,v}^1(h, h))^2.$$

Then we can have a new bound for the scalar product used before

$$(3.15) \quad \frac{d}{dt} \langle \nabla_x h, \nabla_v h \rangle_{L_{x,v}^2} \leq \frac{K_1^\perp e}{\varepsilon^3} \|\nabla_x h^\perp\|_\Lambda^2 + \frac{1}{4C_{\pi 1} C_\pi C_p e \varepsilon} \|\nabla_v h^\perp\|_\Lambda^2 - \frac{1}{2\varepsilon} \|\nabla_x h\|_{L_{x,v}^2}^2 + \frac{4C_\pi}{\varepsilon} (\mathcal{G}_{x,v}^1(h, h))^2,$$

where e is any number greater than 1.

As usual, we may need the same kind of bounds in higher degree Sobolev spaces. The reader may notice that the bounds we are about to write are more intricate than the ones in the previous section because they involve more terms with less derivatives. We consider multi-indexes j and l such that $|j| + |l| = k$. This time we really have to divide in two different cases.

Firstly when $|j| \geq 2$,

$$\begin{aligned}
\frac{d}{dt} \|\partial_l^j h^\perp\|_{L_{x,v}^2}^2 &\leq -\frac{\nu_5^\Lambda}{\varepsilon^2} \|\partial_l^j h^\perp\|_\Lambda^2 + \frac{9(\nu_1^\Lambda)^2 N}{2(\nu_0^\Lambda)^2 \nu_5^\Lambda} \sum_{i, c_i(j) > 0} \|\partial_{l+\delta_i}^{j-\delta_i} h^\perp\|_\Lambda^2 \\
(3.16) \quad &+ K_{dl}^\perp \sum_{|l'| \leq k-1} \|\partial_{l'}^0 h\|_{L_{x,v}^2}^2 + \frac{K_{k-1}^\perp}{\varepsilon^2} \|h^\perp\|_{H_{x,v}^{k-1}}^2 + \frac{3}{\nu_5^\Lambda} (\mathcal{G}_{x,v}^k(h, h))^2.
\end{aligned}$$

Then the case when $|j| = 1$

$$\begin{aligned}
\frac{d}{dt} \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_{L_{x,v}^2}^2 &\leq -\frac{\nu_5^\Lambda}{\varepsilon^2} \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_\Lambda^2 + K_{dl}^\perp \sum_{|l'|=k} \|\partial_{l'}^0 h\|_{L_{x,v}^2}^2 + \frac{K_{k-1}^\perp}{\varepsilon^2} \|h^\perp\|_{H_{x,v}^{k-1}}^2 \\
(3.17) \quad &+ \frac{3}{\nu_5^\Lambda} (\mathcal{G}_{x,v}^k(h, h))^2.
\end{aligned}$$

Finally we give a new version of the control over the scalar product in higher Sobolev's spaces.

$$\begin{aligned}
\frac{d}{dt} \langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2} &\leq \frac{\tilde{K}^\perp}{\varepsilon^3} e \|\partial_l^0 h^\perp\|_\Lambda^2 + \frac{1}{4C_{\pi k} C_\pi N e \varepsilon} \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_\Lambda^2 - \frac{1}{2\varepsilon} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \\
(3.18) \quad &+ \frac{1}{4N\varepsilon} \sum_{|l'| \leq k-1} \|\partial_{l'}^0 h\|_{L_{x,v}^2}^2 + \frac{2C_\pi}{\varepsilon} (\mathcal{G}_{x,v}^k(h, h))^2,
\end{aligned}$$

for any $e \geq 1$.

4. LINEAR CASE: PROOF OF THEOREM 2.1

In this section we are looking at the linear equation

$$\partial_t h = G_\varepsilon(h), \text{ on } \mathbb{T}^N \times \mathbb{R}^N.$$

Theorem 2.1 will be proved by induction on k . We remind here the operator we will work with on $H_{x,v}^k$

- in the case $k = 1$:

$$\|h\|_{\mathcal{H}_\varepsilon^1}^2 = A \|h\|_{L_{x,v}^2}^2 + \alpha \|\nabla_x h\|_{L_{x,v}^2}^2 + b\varepsilon^2 \|\nabla_v h\|_{L_{x,v}^2}^2 + a\varepsilon \langle \nabla_x h, \nabla_v h \rangle_{L_{x,v}^2},$$

- in the case $k > 1$:

$$\|h\|_{\mathcal{H}_\varepsilon^k}^2 = \sum_{\substack{|j|+|l| \leq k \\ |j| \geq 1}} b_{j,l}^{(k)} \varepsilon^2 \|\partial_l^j h\|_{L_{x,v}^2}^2 + \sum_{|l| \leq k} \alpha_l^{(k)} \|\partial_l^0 h\|_{L_{x,v}^2}^2 + \sum_{\substack{|l| \leq k \\ i, c_i(l) > 0}} a_{i,l}^{(k)} \varepsilon \langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2}.$$

The Theorem 2.1 only requires us to choose suitable coefficients that gives us the expected inequality and equivalence.

Consider h_{in} in $H_{x,v}^k \cap \text{Dom}(G_\varepsilon)$. Let h be a solution of $\partial_t h = G_\varepsilon(h)$ on $\mathbb{T}^N \times \mathbb{R}^N$ such that $h(0, \cdot, \cdot) = h_{in}(\cdot, \cdot)$.

Notice that if h_{in} is in $H_{x,v}^k \cap \text{Dom}(G_\varepsilon) \cap \text{Ker}(G_\varepsilon)$ then we have that the associated solution remains the same in time : $\partial_t h = 0$. Therefore the fluid part of a solution does not evolve in time and so the semigroup is identity on $\text{Ker}(G_\varepsilon)$. Besides, we can see directly from the definition and the adjointness property of L that $h \in \text{Ker}(G_\varepsilon)^\perp$ for all t if h_{in} belongs in $\text{Ker}(G_\varepsilon)^\perp$.

Therefore, to prove the theorem it is enough to consider h_{in} in $H_{x,v}^k \cap \text{Dom}(G_\varepsilon) \cap \text{Ker}(G_\varepsilon)^\perp$.

4.1. The case $k = 1$. For now on we assume that our operator L satisfies the conditions (H1), (H2) and (H3) and that $0 < \varepsilon \leq 1$.

If (H3) holds for L then we have that $\varepsilon^{-2}L$ is a negative self-adjoint operator on $L_{x,v}^2$. Moreover, $\varepsilon^{-1}v \cdot \nabla_x$ is skew-symmetric on $L_{x,v}^2$. Therefore it is straightforward to deduce that G_ε yields a C^0 -semigroup on $L_{x,v}^2$ for all positive ε .

Using the toolbox, which is possible since h is in $\text{Ker}(G_\varepsilon)^\perp$ for all t , we just have to consider the linear combination $A(3.6) + \alpha(3.7) + b\varepsilon^2(3.8) + a\varepsilon(3.9)$ to raise

$$(4.1) \quad \begin{aligned} \frac{d}{dt} \|h\|_{\mathcal{H}_\varepsilon^1}^2 &\leq \frac{1}{\varepsilon^2} [bK_1 - \lambda A] \|h^\perp\|_\Lambda^2 + \frac{1}{\varepsilon^2} [C^L e a - \lambda \alpha] \|\nabla_x h^\perp\|_\Lambda^2 \\ &\quad + \left[\frac{2C^L a}{e} - b\nu_3^\Lambda \right] \|\nabla_v h\|_\Lambda^2 + [bK_{dx} - a] \|\nabla_x h\|_{L_{x,v}^2}^2. \end{aligned}$$

Then we make the following decisions:

- (1) We fix b such that $-\nu_3^\Lambda b < -1$.
- (2) We fix A big enough such that $[bK_1 - \lambda A] \leq -1$.
- (3) We fix a big enough such that $[bK_{dx} - a] \leq -1$.
- (4) We fix e big enough such that $\left[\frac{2C^L a}{e} - b\nu_3^\Lambda \right] \leq -1$.
- (5) We fix α big enough such that $[C^L e a - \lambda \alpha] \leq -1$ and such that $\begin{cases} a^2 &\leq \alpha b \\ b &\leq \alpha \end{cases}$.

This leads to, because $0 < \varepsilon \leq 1$:

$$\frac{d}{dt} \|h\|_{\mathcal{H}_\varepsilon^1}^2 \leq - \left(\|h^\perp\|_\Lambda^2 + \|\nabla_x h^\perp\|_\Lambda^2 + \|\nabla_v h\|_\Lambda^2 + \|\nabla_x h\|_{L_{x,v}^2}^2 \right).$$

Finally we can apply the Poincaré inequality (3.5) together with the equivalence of the $L_{x,v}^2$ -norm and the Λ -norm on the fluid part π_L , equation (3.4), to get

$$\exists C, C' > 0, \quad \begin{cases} \|h\|_\Lambda^2 &\leq C \left(\|h^\perp\|_\Lambda^2 + \frac{1}{2} \|\nabla_x h\|_{L_{x,v}^2}^2 \right), \\ \|\nabla_x h\|_\Lambda^2 &\leq C' \left(\|\nabla_x h^\perp\|_\Lambda^2 + \frac{1}{2} \|\nabla_x h\|_{L_{x,v}^2}^2 \right). \end{cases}$$

Therefore we proved the following result:

$$\exists K > 0, \forall 0 < \varepsilon \leq 1, \quad \frac{d}{dt} \|h\|_{\mathcal{H}_\varepsilon^1}^2 \leq -C_G^{(1)} \left(\|h\|_\Lambda^2 + \|\nabla_{x,v} h\|_\Lambda^2 \right).$$

With these constants, $\|\cdot\|_{\mathcal{H}_\varepsilon^1}$ is equivalent to

$$\left(\|h\|_{L_{x,v}^2}^2 + \|\nabla_x h\|_{L_{x,v}^2}^2 + \varepsilon^2 \|\nabla_v h\|_{L_{x,v}^2}^2 \right)^{1/2}$$

since $a^2 \leq \alpha b$ and $b \leq \alpha$ and hence:

$$A \|h\|_{L_{x,v}^2}^2 + \frac{b}{2} \left(\|\nabla_x h\|_{L_{x,v}^2}^2 + \varepsilon^2 \|\nabla_v h\|_{L_{x,v}^2}^2 \right) \leq \|h\|_{\mathcal{H}_\varepsilon^1}^2$$

and

$$\|h\|_{\mathcal{H}_\varepsilon^1}^2 \leq A \|h\|_{L_{x,v}^2}^2 + \frac{3\alpha}{2} \left(\|\nabla_x h\|_{L_{x,v}^2}^2 + \varepsilon^2 \|\nabla_v h\|_{L_{x,v}^2}^2 \right).$$

The results above gives us the expected theorem for $k = 1$.

4.2. The induction in higher order Sobolev spaces. Then we assume that the theorem is true up to the integer $k - 1$, $k > 1$. Then we suppose that L satisfies (H1'), (H2') and (H3) and we consider ε in $(0, 1]$.

Let h_{in} be in $H_{x,v}^k \cap \text{Dom}(G_\varepsilon) \cap \text{Ker}(G_\varepsilon)^\perp$ and h be the solution of $\partial_t h = G_\varepsilon(h)$ such that $h(0, \cdot, \cdot) = h_{in}(\cdot, \cdot)$.

As before, h belongs to $\text{Ker}(G_\varepsilon)^\perp$ for all t and thus we can use the results given by the toolbox.

Thanks to the proof in the case $k = 1$ we know that we are able to handle the case where there is only a difference of one derivative between the number of derivatives in x and in v . Therefore, instead of working with the entire norm of $H_{x,v}^k$, we will look at an equivalent of the Sobolev semi-norm. We define:

$$\begin{aligned} F_k(t) &= \sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} \varepsilon^2 B \|\partial_l^j h\|_{L_{x,v}^2}^2 + B' \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} Q_{l,i}(t), \\ Q_{l,i}(t) &= \alpha \|\partial_l^0 h\|_{L_{x,v}^2}^2 + b\varepsilon^2 \|\partial_{l-\delta_i}^{\delta_i} h\|_{L_{x,v}^2}^2 + a\varepsilon \langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2}, \end{aligned}$$

where the constants, strictly positive, will be chosen later.

Like in the section above, we shall study the time evolution of every term involved in F_k in order to bound above $dF_k/dt(t)$ with negative coefficients.

4.2.1. The time evolution of $Q_{l,i}$. We will first study the time evolution of $Q_{l,i}$ for given $|j| + |l| = k$. The toolbox already gave us all the bounds we need and we just have to gather them in the following way: $\alpha(3.10) + b\varepsilon^2(3.12) + a\varepsilon(3.13)$. This raises, because $0 < \varepsilon \leq 1$,

$$\begin{aligned} \frac{d}{dt} Q_{l,i}(t) &\leq \frac{1}{\varepsilon^2} [C^L e a - \lambda \alpha] \|\partial_l^0 h^\perp\|_\Lambda^2 + \left[\frac{2C^L a}{e} - \nu_5^\Lambda b \right] \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2 \\ &\quad + \left[\frac{3\nu_1^\Lambda}{\nu_5^\Lambda \nu_0^\Lambda} b - a \right] \|\partial_l^0 h\|_{L_{x,v}^2}^2 + K_{k-1} b \|h\|_{H_{x,v}^{k-1}}. \end{aligned}$$

One can notice that, except for the last term, we have exactly the same kind of bound as in (4.1), in the proof of the case $k = 1$. Therefore we can choose α , b , a , e , independently of ε such that it exists $K_Q > 0$ and $C_{k-1} > 0$ such that for all $0 < \varepsilon \leq 1$:

- $Q_{l,i}(t) \sim \|\partial_l^0 h\|_{L_{x,v}^2}^2 + \varepsilon^2 \|\partial_{l-\delta_i}^{\delta_i} h\|_{L_{x,v}^2}^2$,
- $\frac{d}{dt} Q_{l,i}(t) \leq -K_Q \left(\|\partial_l^0 h\|_\Lambda^2 + \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2 \right) + C_{k-1} \|h\|_{H_{x,v}^{k-1}}$,

where we used (3.4) (equivalence of norms $L_{x,v}^2$ and Λ on the fluid part) to get

$$\|\partial_l^0 h\|_\Lambda^2 \leq C' \left(\|\partial_l^0 h^\perp\|_\Lambda^2 + \|\partial_l^0 h\|_{L_{x,v}^2}^2 \right).$$

4.2.2. *The time evolution of F_k and conclusion.* The last result about $Q_{l,i}$ gives us that

$$F_k(t) \sim \sum_{|l|=k} \|\partial_l^0 h\|_{L_{x,v}^2}^2 + \varepsilon^2 \sum_{\substack{|l|+|j|=k \\ |j| \geq 1}} \|\partial_l^j h\|_{L_{x,v}^2}^2.$$

To study the time evolution of F_k we just need to combine the evolution of $Q_{l,i}$ and the one of $\|\partial_l^j h\|_{L_{x,v}^2}^2$ which is given in the toolbox by (3.11).

$$(4.2) \quad \begin{aligned} \frac{d}{dt} F_k(t) &\leq \sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} -\nu_5^\Lambda B \|\partial_l^j h\|_\Lambda^2 + \sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} \frac{3(\nu_1^\Lambda)^2 N}{\nu_5^\Lambda (\nu_0^\Lambda)^2} B \varepsilon^2 \sum_{i, c_i(j) > 0} \|\partial_{l+\delta_i}^{j-\delta_i} h\|_\Lambda^2 \\ &\quad - K_Q B' \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} \left(\|\partial_l^0 h\|_\Lambda^2 + \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2 \right) \\ &\quad + \left[\sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} K_{k-1} B + \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} B' C_{k-1} \right] \|h\|_{H_{x,v}^{k-1}}^2. \end{aligned}$$

Then we choose the following coefficients $B = 2/\nu_5^\Lambda$ and we can rearrange the sums to obtain

$$\begin{aligned} \frac{d}{dt} F_k(t) &\leq \sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} \left(\frac{6N(\nu_1^\Lambda)^2}{(\nu_5^\Lambda \nu_0^\Lambda)^2} \varepsilon^2 - 2 \right) \|\partial_l^j h\|_\Lambda^2 + \sum_{\substack{|j|+|l|=k \\ |j|=1}} \left(\frac{6N(\nu_1^\Lambda)^2}{(\nu_5^\Lambda \nu_0^\Lambda)^2} \varepsilon^2 - K_Q B' \right) \|\partial_l^j h\|_\Lambda^2 \\ &\quad + \sum_{\substack{|j|+|l|=k \\ |j|=0}} (-K_Q B') \|\partial_l^j h\|_\Lambda^2 + C_+^{(k-1)}(B') \|h\|_{H_{x,v}^{k-1}}^2. \end{aligned}$$

Therefore we can choose the remaining coefficients:

- (1) $\varepsilon_N = \min \left\{ 1, \sqrt{\frac{(\nu_5^\Lambda \nu_0^\Lambda)^2}{6N(\nu_1^\Lambda)^2}} \right\},$
- (2) we fix B' big enough such that $K_Q B' \geq 1$ and $\left(\frac{6N(\nu_1^\Lambda)^2}{(\nu_5^\Lambda \nu_0^\Lambda)^2} \varepsilon_N^2 - K_Q B' \right) \leq -1.$

Everything is now fixed in $C_+^{(k-1)}(B')$ and therefore it is just a constant $C_+^{(k-1)}$ that does not depend on ε . Therefore we then have the final result.

$$\forall 0 < \varepsilon \leq \varepsilon_N, \quad \frac{d}{dt} F_k(t) \leq C_+^{(k-1)} \|h\|_{H_{x,v}^{k-1}}^2 - \left(\sum_{|j|+|l|=k} \|\partial_l^j h\|_\Lambda^2 \right).$$

Then, we know that $\|\cdot\|_\Lambda$ controls the L^2 -norm. And therefore:

$$\forall 0 < \varepsilon \leq \varepsilon_N, \quad \frac{d}{dt} F_k(t) \leq C_+^{(k)} \left(\sum_{|j|+|l| \leq k-1} \|\partial_l^j h\|_\Lambda^2 \right) - \left(\sum_{|j|+|l|=k} \|\partial_l^j h\|_\Lambda^2 \right).$$

This inequality is true for all k and therefore we can take a linear combination of the F_k to obtain the following, where C_k is a constant that does not depend on ε since $C_+^{(k)}$ does not depend on it.

$$\forall 0 < \varepsilon \leq \varepsilon_N, \quad \frac{d}{dt} \left(\sum_{p=1}^n C_p F_p(t) \right) \leq -C_G^{(k)} \left(\sum_{|j|+|l| \leq k} \|\partial_l^j h\|_\Lambda^2 \right).$$

We can use the induction assumption from rank 1 up to rank $k-1$ to find that this linear combination is equivalent to

$$\|\cdot\|_{L_{x,v}^2}^2 + \sum_{|l| \leq k} \|\partial_l^0 \cdot\|_{L_{x,v}^2}^2 + \varepsilon^2 \sum_{\substack{|l|+|j| \leq k \\ |j| \geq 1}} \|\partial_l^j \cdot\|_{L_{x,v}^2}^2$$

and so fits the expected requirements.

5. ESTIMATE FOR THE FULL EQUATION: PROOF OF PROPOSITION 2.2

We will prove that proposition by induction on k . For now on we assume that L satisfies hypothesis (H1'), (H2') and (H3), that Γ satisfies properties (H4) and (H5) and we take g in $H_{x,v}^k$.

So we take h_{in} in $H_{x,v}^k \cap \text{Ker}(G_\varepsilon)^\perp$ and we consider the associated solution, denoted by h , of

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon} \Gamma(g, h).$$

One can notice that thanks to (H5) and the self-adjointness of L , h remains in $\text{Ker}(G_\varepsilon)^\perp$ for all times.

Besides, while considering the time evolution we find a term due to G_ε and another due to Γ . Therefore, we will use the results found in the toolbox but including the terms in parenthesis.

5.1. The case $k = 1$. We want to study the following operator on $H_{x,v}^k$

$$\|h\|_{\mathcal{H}_\varepsilon^1}^2 = A \|h\|_{L_{x,v}^2}^2 + \alpha \|\nabla_x h\|_{L_{x,v}^2}^2 + b\varepsilon^2 \|\nabla_v h\|_{L_{x,v}^2}^2 + a\varepsilon \langle \nabla_x h, \nabla_v h \rangle_{L_{x,v}^2}.$$

Therefore, using the toolbox we just have to consider the linear combination $A(3.6) + \alpha(3.7) + b\varepsilon^2(3.8) + a\varepsilon(3.9)$ to raise

$$\begin{aligned} \frac{d}{dt} \|h\|_{\mathcal{H}_\varepsilon^1}^2 &\leq \frac{1}{\varepsilon^2} [bK_1 - \lambda A] \|h^\perp\|_\Lambda^2 + \frac{1}{\varepsilon^2} [C^L e a - \lambda \alpha] \|\nabla_x h^\perp\|_\Lambda^2 \\ (5.1) \quad &+ \left[\frac{2C^L a}{e} - b\nu_3^\Lambda \right] \|\nabla_v h\|_\Lambda^2 + [bK_{dx} - a] \|\nabla_x h\|_{L_{x,v}^2}^2 \\ &+ \frac{A\nu_1^\Lambda}{\nu_0^\Lambda \lambda} (\mathcal{G}_x^0(g, h))^2 + \left[\frac{\alpha\nu_1^\Lambda}{\nu_0^\Lambda \lambda} + \frac{\nu_1^\Lambda e a}{C^L \nu_0^\Lambda} \right] (\mathcal{G}_x^1(g, h))^2 \\ &+ \frac{3\nu_1^\Lambda b}{\nu_0^\Lambda \nu_3^\Lambda} \varepsilon^2 (\mathcal{G}_{x,v}^1(g, h))^2. \end{aligned}$$

One can see that we obtained exactly the same upper bound as in the proof of the previous theorem, equation (4.1), adding the terms involving Γ (remember that \mathcal{G}_x^k is increasing in k). Therefore we can make the same choices for A , α , b , a and e , independently of Γ and g , to get that

$$\|h\|_{\mathcal{H}_\varepsilon^1}^2 \sim \|h\|_{L_{x,v}^2}^2 + \|\nabla_x h\|_{L_{x,v}^2}^2 + \varepsilon^2 \|\nabla_v h\|_{L_{x,v}^2}^2,$$

and that, once those parameters are fixed, there exist $K_0^{(1)}$, $K_1^{(1)}$, $K_2^{(1)} > 0$ such that for all $0 < \varepsilon \leq 1$,

$$\frac{d}{dt} \|h\|_{\mathcal{H}_\varepsilon^1}^2 \leq -K_0^{(1)} (\|h\|_\Lambda^2 + \|\nabla_{x,v} h\|_\Lambda^2) + K_1^{(1)} (\mathcal{G}_x^1(g, h))^2 + \varepsilon^2 K_2^{(1)} (\mathcal{G}_{x,v}^1(g, h))^2,$$

which is the expected result in the case $k = 1$.

5.2. The induction in higher order Sobolev spaces. Then we assume that the theorem is true up to the integer $k - 1$, $k > 1$. Then we suppose that L satisfies (H1'), (H2') and (H3) and we consider ε in $(0, 1]$.

Since h_{in} is in $\text{Ker}(G_\varepsilon)^\perp$, h belongs to $\text{Ker}(G_\varepsilon)^\perp$ for all t and so we can use the results given in the toolbox.

As in the proof in the linear case we define:

$$\begin{aligned} F_k(t) &= \sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} \varepsilon^2 B \|\partial_l^j h\|_{L_{x,v}^2}^2 + B' \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} Q_{l,i}(t), \\ Q_{l,i}(t) &= \alpha \|\partial_l^0 h\|_{L_{x,v}^2}^2 + b\varepsilon^2 \|\partial_{l-\delta_i}^{\delta_i} h\|_{L_{x,v}^2}^2 + a\varepsilon \langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2}, \end{aligned}$$

where the constants, strictly positive, will be chosen later.

Like in the section above, we shall study the time evolution of every term involved in F_k in order to bound above $dF_k/dt(t)$ with expected coefficients.

5.2.1. The time evolution of $Q_{l,i}$. We will first study the time evolution of $Q_{l,i}$ for given $|j| + |l| = k$. The toolbox already gave us all the bounds we need and we just have to gather them in the following way: $\alpha(3.10) + b\varepsilon^2(3.12) + a\varepsilon(3.13)$. This raises, because $0 < \varepsilon \leq 1$,

$$\begin{aligned} \frac{d}{dt} Q_{l,i}(t) &\leq \frac{1}{\varepsilon^2} [C^L e a - \lambda \alpha] \|\partial_l^0 h^\perp\|_\Lambda^2 + \left[\frac{2C^L a}{e} - \nu_5^\Lambda b \right] \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2 \\ &\quad + \left[\frac{3\nu_1^\Lambda}{\nu_5^\Lambda \nu_0^\Lambda} b - a \right] \|\partial_l^0 h\|_{L_{x,v}^2}^2 + K_{k-1} b \|h\|_{H_{x,v}^{k-1}} \\ &\quad + \left[\frac{\alpha \nu_1^\Lambda}{\nu_0^\Lambda \lambda} + \frac{\nu_1^\Lambda e a}{C^L \nu_0^\Lambda} \right] (\mathcal{G}_x^k(g, h))^2 + \frac{3\nu_1^\Lambda b}{\nu_0^\Lambda \nu_5^\Lambda} \varepsilon^2 (\mathcal{G}_{x,v}^k(g, h))^2. \end{aligned}$$

One can notice that, except for the term in $\|h\|_{H_{x,v}^{k-1}}$, we have exactly the same kind of bound as in the case $k = 1$, given by (5.1). Therefore we can choose α , b , a , e , independently of ε , Γ and g such that it exists K_Q , $K_{\Gamma 1}$, $K_{\Gamma 2} > 0$ and $C_{k-1} > 0$ such that for all $0 < \varepsilon \leq 1$:

- $Q_{l,i}(t) \sim \|\partial_l^0 h\|_{L_{x,v}^2}^2 + \varepsilon^2 \|\partial_{l-\delta_i}^{\delta_i} h\|_{L_{x,v}^2}^2,$

•

$$\begin{aligned} \frac{d}{dt} Q_{l,i}(t) &\leq -K_Q \left(\|\partial_l^0 h\|_\Lambda^2 + \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2 \right) + K_{\Gamma 1} (\mathcal{G}_x^k(g, h))^2 \\ &\quad + \varepsilon^2 K_{\Gamma 2} (\mathcal{G}_{x,v}^k(g, h))^2 + C_{k-1} \|h\|_{H_{x,v}^{k-1}}, \end{aligned}$$

where we used (3.4) (equivalence of norms $L_{x,v}^2$ and Λ on the fluid part) to get

$$\|\partial_l^0 h\|_\Lambda^2 \leq C' \left(\|\partial_l^0 h^\perp\|_\Lambda^2 + \|\partial_l^0 h\|_{L_{x,v}^2}^2 \right).$$

5.2.2. *The time evolution of F_k and conclusion.* The last result about $Q_{l,i}$ gives us that

$$F_k(t) \sim \sum_{|l|=k} \|\partial_l^0 h\|_{L_{x,v}^2}^2 + \varepsilon^2 \sum_{\substack{|l|+|j|=k \\ |j| \geq 1}} \|\partial_l^j h\|_{L_{x,v}^2}^2,$$

so it remains to show that F_k satisfies the property describe by the theorem for some B and B' .

To study the time evolution of F_k we just need to combine the evolution of $Q_{l,i}$ and the one of $\|\partial_l^j h\|_{L_{x,v}^2}^2$ which is given in the toolbox by (3.11).

$$\begin{aligned} \frac{d}{dt} F_k(t) &\leq \sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} -\nu_5^\Lambda B \|\partial_l^j h\|_\Lambda^2 + \sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} \frac{3(\nu_1^\Lambda)^2 N}{\nu_5^\Lambda (\nu_0^\Lambda)^2} B \varepsilon^2 \sum_{i, c_i(j) > 0} \left\| \partial_{l+\delta_i}^{j-\delta_i} h \right\|_\Lambda^2 \\ &\quad - K_Q B' \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} \left(\|\partial_l^0 h\|_\Lambda^2 + \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2 \right) \\ (5.2) \quad &+ \left[\sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} K_{k-1} B + \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} B' C_{k-1} \right] \|h\|_{H_{x,v}^{k-1}}^2 \\ &+ \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} B' K_{\Gamma 1} (\mathcal{G}_x^k(g, h))^2 \\ &+ \varepsilon^2 \left[\sum_{\substack{|l|=k \\ i, c_i(l) > 0}} B' K_{\Gamma 2} + \sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} \frac{3\nu_1^\Lambda}{\nu_0^\Lambda \nu_5^\Lambda} B \right] (\mathcal{G}_{x,v}^k(g, h))^2. \end{aligned}$$

One can easily see that, apart from the terms including Γ , we have exactly the same bound as in the proof in the linear case, equation (4.2). Therefore we can choose B , B' and ε_N like we did, thus independent of Γ and g , to have for all $0 < \varepsilon \leq \varepsilon_N$

$$\begin{aligned} \frac{d}{dt} F_k(t) &\leq C_+^{(k-1)} \|h\|_{H_{x,v}^{k-1}}^2 - \left(\sum_{|j|+|l|=k} \|\partial_l^j h\|_\Lambda^2 \right) \\ &\quad + \tilde{K}_{\Gamma 1} (\mathcal{G}_x^k(g, h))^2 + \varepsilon^2 \tilde{K}_{\Gamma 2} (\mathcal{G}_{x,v}^k(g, h))^2, \end{aligned}$$

with $C_+^{(k-1)}$, \tilde{K}_{Γ_1} and \tilde{K}_{Γ_2} positive constants independent of ε , Γ and g .

To conclude we just have to, as in the linear case, take a linear combination of the $(F_p)_{p \leq k}$ and use the induction hypothesis (remember that both $\mathcal{G}_{x,v}^p$ and \mathcal{G}_x^p are increasing functions of p) to obtain the expected result: $\forall 0 < \varepsilon \leq \varepsilon_N$,

$$\begin{aligned} \frac{d}{dt} \left(\sum_{p=1}^n C_p F_p(t) \right) \leq & - K_0^{(k)} \left(\sum_{|j|+|l| \leq k} \|\partial_t^j h\|_{\Lambda}^2 \right) + K_1^{(k)} (\mathcal{G}_x^k(g, h))^2 \\ & + \varepsilon^2 K_1^{(k)} (\mathcal{G}_{x,v}^k(g, h))^2, \end{aligned}$$

with this linear combination being equivalent to

$$\|\cdot\|_{L_{x,v}^2}^2 + \sum_{|l| \leq k} \|\partial_t^0 \cdot\|_{L_{x,v}^2}^2 + \varepsilon^2 \sum_{\substack{|l|+|j| \leq k \\ |j| \geq 1}} \|\partial_t^j \cdot\|_{L_{x,v}^2}^2$$

and so fits the expected requirements.

6. EXISTENCE AND EXPONENTIAL DECAY: PROOF OF THEOREM 2.3

One can clearly see that solving the kinetic equation (1.2) in the setting $f = M + \varepsilon M^{1/2} h$ is equivalent to solving the linearized kinetic equation (1.3) directly. Therefore we are going to focus only on this linearized equation.

We will do the proof in two steps. First we will prove the existence of global solutions to the linearized equation on condition that the initial data is close enough to the equilibrium. Then we will prove that if we have a global solution to the linearized equation then we have an exponential decay towards 0 for the perturbation as long as the initial perturbation is small enough. Finally, combining those two results gives us the proof of the theorem, taking an initial perturbation small enough to satisfy each of the two conditions required above.

The issue of the positivity of the solution is easy once we have the first two steps of our proof. Indeed, it will be enough to consider the initial perturbation to be, in $\mathcal{H}_\varepsilon^k$, smaller than the global equilibrium since $k > N/2$ and so $L_{x,v}^\infty$ is embedded in $H_{x,v}^k$. Therefore we will just need a smaller δ_k if we had not a small enough one before.

6.1. Proof of the existence of global solutions.

6.1.1. Construction of solutions to a linearized problem. Here we will follow the general method used in several articles, that is to say trying to approximate our solution by a sequence of solutions of a linearization of our initial problem. Then we have to construct a Sobolev space where this sequence can be uniformly bounded in order to be able to extract a convergent subsequence.

Starting from h_0 in $H_{x,v}^k \cap \text{Ker}(G_\varepsilon)^\perp$, to be define later, we define the function h_{n+1} in $H_{x,v}^k$ by induction on $n \geq 0$:

$$(6.1) \quad \begin{cases} \partial_t h_{n+1} + \frac{1}{\varepsilon} v \cdot \nabla_x h_{n+1} = \frac{1}{\varepsilon^2} L(h_{n+1}) + \frac{1}{\varepsilon} \Gamma(h_n, h_{n+1}) \\ h_{n+1}(0, x, v) = h_{in}(x, v), \end{cases}$$

First we need to check if our sequence is well-defined.

Lemma 6.1. *Let L be satisfying assumptions (H1'), (H2') and (H3), and let Γ be satisfying assumptions (H4) and (H5).*

Then, it exists $0 < \varepsilon_N \leq 1$ such that for all $k \geq k_0$ (defined in (H4)), it exists $\delta_k > 0$ such that for all $0 < \varepsilon \leq \varepsilon_N$, if $\|h_{in}\|_{\mathcal{H}_\varepsilon^k} \leq \delta_k$ then the sequence $(h_n)_{n \in \mathbb{N}}$ is well-defined, continuous in time, in $H_{x,v}^k$ and belongs to $\text{Ker}(G_\varepsilon)^\perp$.

Proof of Lemma 6.1. By induction, let us suppose that for a fixed $n \geq 0$ we have constructed h_n in $H_{x,v}^k$, which is true for h_{in} .

Using the previous notation one can see that we are in fact trying to solve the linear equation on the torus:

$$\partial_t h_{n+1} = G_\varepsilon(h_{n+1}) + \frac{1}{\varepsilon} \Gamma(h_n, h_{n+1})$$

with h_{in} as an initial data.

The existence of a solution h_{n+1} has already been shown for each equation covered by the hypo-coercivity theory in the case $\varepsilon = 1$ (see papers described in the introduction). It was proved by fixed point arguments applied to the Duhamel's formula. In order not to write several times the same estimates one may use our next lemma 6.2 together with the Duhamel's formula (instead of considering directly the time derivative of h_{n+1}) to get a fixed point argument as long as h_{in} is small enough, the smallness not depending on ε .

As shown in the study of the linear part of the linearized model, under assumptions (H1'), (H2') and (H3) G_ε generates a C^0 -semigroup on $H_{x,v}^k$, for all $0 < \varepsilon \leq \varepsilon_N$. Moreover, hypothesis (H4) shows us that $\Gamma(h_n, \cdot)$ is a linear operator from $H_{x,v}^k$ to $H_{x,v}^k$. Thus h_{n+1} is in $H_{x,v}^k$.

The belonging to $\text{Ker}(G_\varepsilon)^\perp$ is direct since $\Gamma(h_n, \cdot)$ is in $\text{Ker}(G_\varepsilon)^\perp$ (hypothesis (H5)) and the fact that this space is stable under the action of G_ε . \square

Then we have to strongly bound the sequence, at least in short time, to have a chance to obtain a convergent subsequence, up to an extraction.

6.1.2. Boundedness of the sequence. We are about to prove the global existence in time on solution in $C(\mathbb{R}^+, \|\cdot\|_{\mathcal{H}_\varepsilon^k})$. That will give us existence of solutions in standard Sobolev's spaces as long as the initial data is small enough in the sense of the $\mathcal{H}_\varepsilon^k$ -norm, which is smaller than the standard $H_{x,v}^k$ -norm. To achieve that we define a new norm on $H_{x,v}^k$

$$E(h) = \sup_{t \in \mathbb{R}^+} \left(\|h(t)\|_{\mathcal{H}_\varepsilon^k}^2 + \int_0^t \|h(s)\|_{H_\Lambda^k}^2 ds \right).$$

Lemma 6.2. *Let L be satisfying assumptions (H1'), (H2') and (H3), and let Γ be satisfying assumptions (H4) and (H5).*

Then it exists $0 < \varepsilon_N \leq 1$ such that for all $k \geq k_0$ (defined in (H4)) it exists $\delta_k > 0$ independent of ε , such that for all $0 < \varepsilon \leq \varepsilon_N$, if $\|h_{in}\|_{\mathcal{H}_\varepsilon^k} \leq \delta_k$ then

$$(E(h_n) \leq \delta_k) \Rightarrow (E(h_{n+1}) \leq \delta_k).$$

Proof of Lemma 6.2. We let $t > 0$.

We know that h_{in} belongs to $H_{x,v}^k \cap \text{Ker}(G_\varepsilon)^\perp$. Moreover we have, thanks to Lemma 6.1, that (h_n) is well-defined, in $\text{Ker}(G_\varepsilon)^\perp$ and in $H_{x,v}^k$, since $k \geq k_0$. Moreover, Γ satisfies (H5). Therefore we can use the Proposition 2.2 to write, for $\varepsilon \leq \varepsilon_N$ (ε_N being the minimum between the one in Lemma 6.1 and the one in Proposition 2.2),

$$\frac{d}{dt} \|h_{n+1}\|_{\mathcal{H}_\varepsilon^k}^2 \leq -K_0^{(k)} \|h_{n+1}\|_{H_\Lambda^k}^2 + K_1^{(k)} (\mathcal{G}_x^k(h_n, h_{n+1}))^2 + \varepsilon^2 K_2^{(k)} (\mathcal{G}_{x,v}^k(h_n, h_{n+1}))^2.$$

We can use the hypothesis (H4) and the fact that

$$(6.2) \quad C_m \left(\|\cdot\|_{L_{x,v}^2}^2 + \sum_{|l| \leq k} \|\partial_l^0 \cdot\|_{L_{x,v}^2}^2 + \varepsilon^2 \sum_{\substack{|l|+|j| \leq k \\ |j| \geq 1}} \|\partial_l^j \cdot\|_{L_{x,v}^2}^2 \right) \leq \|\cdot\|_{\mathcal{H}_\varepsilon^k}^2 \leq C_M \|\cdot\|_{H_{x,v}^k},$$

to get the following upper bounds:

$$\begin{aligned} (\mathcal{G}_x^k(h_n, h_{n+1}))^2 &\leq \frac{C_\Gamma^2}{C_m} \left(\|h_n\|_{\mathcal{H}_\varepsilon^k}^2 \|h_{n+1}\|_{H_\Lambda^k}^2 + \|h_{n+1}\|_{\mathcal{H}_\varepsilon^k}^2 \|h_n\|_{H_\Lambda^k}^2 \right) \\ (\mathcal{G}_{x,v}^k(h_n, h_{n+1}))^2 &\leq \frac{C_\Gamma^2}{C_m \varepsilon^2} \left(\|h_n\|_{\mathcal{H}_\varepsilon^k}^2 \|h_{n+1}\|_{H_\Lambda^k}^2 + \|h_{n+1}\|_{\mathcal{H}_\varepsilon^k}^2 \|h_n\|_{H_\Lambda^k}^2 \right). \end{aligned}$$

Therefore we have the following upper bound, where K_1 and K_2 are constants independent of ε :

$$\begin{aligned} \frac{d}{dt} \|h_{n+1}\|_{\mathcal{H}_\varepsilon^k}^2 &\leq -K_0^{(k)} \|h_{n+1}\|_{H_\Lambda^k}^2 + K_1 \|h_n\|_{\mathcal{H}_\varepsilon^k}^2 \|h_{n+1}\|_{H_\Lambda^k}^2 + K_2 \|h_{n+1}\|_{\mathcal{H}_\varepsilon^k}^2 \|h_n\|_{H_\Lambda^k}^2 \\ &\leq \left[K_1 E(h_n) - K_0^{(k)} \right] \|h_{n+1}\|_{H_\Lambda^k}^2 + K_2 E(h_{n+1}) \|h_n\|_{H_\Lambda^k}^2. \end{aligned}$$

We consider now that $E(h_n) \leq K_0^{(k)}/2K_1$.

We can integrate the equation above between 0 and t and one obtains

$$\|h_{n+1}\|_{\mathcal{H}_\varepsilon^k}^2 + \frac{K_0^{(k)}}{2} \int_0^t \|h_{n+1}\|_{H_\Lambda^k}^2 ds \leq \|h_0\|_{\mathcal{H}_\varepsilon^k}^2 + K E(h_{n+1}) E(h_n).$$

This is true for all $t > 0$, then we define $C = \min\{1, K_0^{(k)}/2\}$, if $E(h_n) \leq C/2K$ we have

$$E(h_{n+1}) \leq \frac{2}{C} \|h_0\|_{\mathcal{H}_\varepsilon^k}^2.$$

Therefore choosing $M^{(k)} = \min\{C/2K, K_0^{(k)}/2K_1\}$ and $\delta_k \leq \min\{M^{(k)}C/2, M^{(k)}\}$ gives us the expected result. \square

6.1.3. *The global existence of solutions.* Now we are able to prove the global existence result:

Theorem 6.3. *Let L be satisfying assumptions (H1'), (H2') and (H3), and let Γ be satisfying assumptions (H4) and (H5).*

Then it exists $0 < \varepsilon_N \leq 1$ such that for all $k \geq k_0$ (defined in (H4)), it exists $\delta_k > 0$ and for all $0 < \varepsilon \leq \varepsilon_N$:

If $\|h_{in}\|_{\mathcal{H}_\varepsilon^k} \leq \delta_k$ then there exist a solution of (1.3) in $C(\mathbb{R}^+, E(\cdot))$.

Proof of Theorem 6.3. Regarding Lemma 6.2, by induction we can strongly bound the sequence $(h_n)_{n \in \mathbb{N}}$, as long as $E(h_0) \leq \delta_k$, constant being defined as in Lemma 6.2. Therefore, defining h_0 to be h_{in} at $t = 0$ and 0 elsewhere gives us $E(h_0) = \|h_{in}\|_{\mathcal{H}_\varepsilon^k} \leq \delta_k$

Thus, we can take the limit in (6.1) as n tends to $+\infty$, since G_ε and Γ are continuous. We obtain h a solution, in $C(\mathbb{R}^+, E(\cdot))$, to

$$\begin{cases} \partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon} \Gamma(h, h) \\ h(0, x, v) = h_{in}(x, v), \end{cases}.$$

□

6.2. Proof of the exponential decay. The function constructed above, h , is in $\text{Ker}(G_\varepsilon)^\perp$ for all $0 < \varepsilon \leq 1$. Moreover, this function is clearly a solution of the following equation:

$$\partial_t h = G_\varepsilon(h) + \frac{1}{\varepsilon} \Gamma(h, h),$$

with Γ satisfying (H5). Therefore, we can use the a priori estimate on solutions of the full perturbative model concerning the time evolution of the $\mathcal{H}_\varepsilon^k$ -norm (where we will omit to write the dependence on k for clearness purpose), Proposition 2.2.

$$\frac{d}{dt} \|h\|_{\mathcal{H}_\varepsilon^k}^2 \leq -K_0 \|h\|_{H_\Lambda^k}^2 + K_1 (\mathcal{G}_x^k(h, h))^2 + \varepsilon^2 K_2 (\mathcal{G}_{x,v}^k(h, h))^2.$$

Moreover, using (6.2) and hypothesis (H4) to find:

$$\begin{aligned} (\mathcal{G}_x^k(h, h))^2 &\leq \frac{2C_\Gamma^2}{C_m} \|h\|_{\mathcal{H}_\varepsilon^k}^2 \|h\|_{H_\Lambda^k}^2 \\ (\mathcal{G}_{x,v}^k(h, h))^2 &\leq \frac{2C_\Gamma^2}{C_m \varepsilon^2} \|h\|_{\mathcal{H}_\varepsilon^k}^2 \|h\|_{H_\Lambda^k}^2. \end{aligned}$$

Hence, K being a constant independent of ε :

$$\frac{d}{dt} \|h\|_{\mathcal{H}_\varepsilon^k}^2 \leq \left(K \|h\|_{\mathcal{H}_\varepsilon^k}^2 - K_0 \right) \|h\|_{H_\Lambda^k}^2.$$

Therefore, one can notice that if $\|h_{in}\|_{\mathcal{H}_\varepsilon^k}^2 \leq K_0/2K$ then we have that $\|h\|_{\mathcal{H}_\varepsilon^k}^2$ is decreasing in time. Hence, because the Λ -norm controls the L^2 -norm which controls the \mathcal{H} -norm:

$$\begin{aligned} \frac{d}{dt} \|h\|_{\mathcal{H}_\varepsilon^k}^2 &\leq -\frac{K_0}{2} \|h\|_{H_\Lambda^k}^2 \\ &\leq -\frac{K_0}{2} \frac{\nu_0^\Lambda}{\nu_1^\Lambda C_M} \|h\|_{\mathcal{H}_\varepsilon^k}^2. \end{aligned}$$

Then we have directly, by Gronwall's lemma and setting $\tau_k = K_0 \nu_0^\Lambda / 4 \nu_1^\Lambda C_M$,

$$\|h\|_{\mathcal{H}_\varepsilon^k}^2 \leq \|h_{in}\|_{\mathcal{H}_\varepsilon^k}^2 e^{-2\tau_k t}$$

as long as $\|h_{in}\|_{\mathcal{H}_\varepsilon^k}^2 \leq K_0/2K$, which is the expected result with $\delta_k \leq \sqrt{K_0/2K}$.

7. EXPONENTIAL DECAY OF v -DERIVATIVES: PROOF OF THEOREM 2.4

In order to prove this theorem we are going to state a proposition giving an a priori estimate on a solution to the equation (1.3)

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon} \Gamma(h, h).$$

We remind the reader that we will work on $H_{x,v}^k$ with the following positive operator

$$\|\cdot\|_{\mathcal{H}_{\varepsilon\perp}^k}^2 = \sum_{\substack{|j|+|l|\leq k \\ |j|\geq 1}} b_{j,l}^{(k)} \|\partial_l^j (Id - \pi_L)\|_{L_{x,v}^2}^2 + \sum_{|l|\leq k} \alpha_l^{(k)} \|\partial_l^0 \cdot\|_{L_{x,v}^2}^2 + \sum_{\substack{|l|\leq k \\ i, c_i(l)>0}} a_{i,l}^{(k)} \varepsilon \langle \partial_{l-\delta_i}^{\delta_i} \cdot, \partial_l^0 \cdot \rangle_{L_{x,v}^2}.$$

One can notice that if we choose coefficients $(b_{j,l}^{(k)})$, $(\alpha_l^{(k)})$, $(a_{i,l}^{(k)}) > 0$ such that $\|\cdot\|_{\mathcal{H}_{1\perp}^k}^2$ is equivalent to

$$\sum_{\substack{|j|+|l|\leq k \\ |j|\geq 1}} \|\partial_l^j (Id - \pi_L)\|_{L_{x,v}^2}^2 + \sum_{|l|\leq k} \|\partial_l^0 \cdot\|_{L_{x,v}^2}^2$$

then for all ε less than some ε_0 , $\|\cdot\|_{\mathcal{H}_{\varepsilon\perp}^k}^2$ is also equivalent to the latter norm with equivalence coefficients not depending on ε .

Moreover, using equation (3.3), we have that

$$\|\partial_l^j h\|_{L_{x,v}^2}^2 \leq C_{\pi k} \|\partial_l^0 h\|_{L_{x,v}^2}^2 + \|\partial_l^j h^\perp\|_{L_{x,v}^2}^2 \leq 2C_{\pi k} \|\partial_l^0 h\|_{L_{x,v}^2}^2 + \|\partial_l^j h\|_{L_{x,v}^2}^2,$$

and therefore

$$\sum_{\substack{|j|+|l|\leq k \\ |j|\geq 1}} \|\partial_l^j (Id - \pi_L)\|_{L_{x,v}^2}^2 + \sum_{|l|\leq k} \|\partial_l^0 \cdot\|_{L_{x,v}^2}^2$$

is equivalent to the standard Sobolev norm. Thus, we will just construct coefficients $(b_{j,l}^{(k)})$, $(\alpha_l^{(k)})$ and $(a_{i,l}^{(k)})$ so that $\|\cdot\|_{\mathcal{H}_{1\perp}^k}^2$ is equivalent to the latter norm and then by taking ε small enough we will have the equivalence, not depending on ε , between $\|\cdot\|_{\mathcal{H}_{\varepsilon\perp}^k}^2$ and the $H_{x,v}^k$ -norm.

7.1. An a priori estimate. In this subsection we will prove the following proposition:

Proposition 7.1. *If L is a linear operator satisfying the conditions (H1'), (H2') and (H3) and Γ a bilinear operator satisfying (H5) then it exists $0 < \varepsilon_N \leq 1$ such that for all k in \mathbb{N}^* ,*

(1) *for h_{in} in $\text{Ker}(G_\varepsilon)^\perp$ if we have h an associated solution of*

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon} \Gamma(h, h),$$

(2) *there exist $K_0^{(k)}, K_1^{(k)}, (b_{j,l}^{(k)}), (\alpha_l^{(k)}), (a_{i,l}^{(k)}) > 0$ such that for all $0 < \varepsilon \leq \varepsilon_N$:*

- $\|\cdot\|_{\mathcal{H}_{\varepsilon^\perp}^k} \sim \|\cdot\|_{H_{x,v}^k},$
- $\forall h_{in} \in H_{x,v}^k \cap \text{Ker}(G_\varepsilon)^\perp,$

$$\frac{d}{dt} \|h\|_{\mathcal{H}_{\varepsilon^\perp}^k}^2 \leq -K_0^{(k)} \left(\frac{1}{\varepsilon^2} \|h^\perp\|_{H_\Lambda^k}^2 + \sum_{1 \leq |l| \leq k} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \right) + K_1^{(k)} (\mathcal{G}_{x,v}^k(h, h))^2.$$

We will prove that proposition by induction on k .

So we take h_{in} in $H_{x,v}^k \cap \text{Ker}(G_\varepsilon)^\perp$ and we consider the associated solution of (1.3), denoted by h . One can notice that thanks to (H5), h remains in $\text{Ker}(G_\varepsilon)^\perp$ for all times and thus we are allowed to use the inequalities given in the toolbox

7.1.1. The case $k = 1$. In that case we have

$$\|h\|_{\mathcal{H}_{\varepsilon^\perp}^1}^2 = A \|h\|_{L_{x,v}^2}^2 + \alpha \|\nabla_x h\|_{L_{x,v}^2}^2 + b \|\nabla_v h^\perp\|_{L_{x,v}^2}^2 + a\varepsilon \langle \nabla_x h, \nabla_v h \rangle_{L_{x,v}^2},$$

with A, α, b and a strictly positive.

Therefore we can study the time evolution of that operator acting on h by gathering results given in the toolbox. We simply take $A(3.6) + \alpha(3.7) + b(3.14) + a\varepsilon(3.15)$

$$\begin{aligned} \frac{d}{dt} \|h\|_{\mathcal{H}_{\varepsilon^\perp}^1}^2 &\leq \frac{1}{\varepsilon^2} [K_1^\perp b - \lambda A] \|h^\perp\|_\Lambda^2 + \frac{1}{\varepsilon^2} [K^\perp e a - \lambda \alpha] \|\nabla_x h^\perp\|_\Lambda^2 \\ &\quad + \frac{1}{\varepsilon^2} \left[\frac{1}{4C_{\pi 1} C_\pi C_p} \frac{a}{e} - b \frac{\nu_3^\Lambda}{2} \right] \|\nabla_v h^\perp\|_\Lambda^2 + \left[K_{dx}^\perp b - \frac{a}{2} \right] \|\nabla_x h\|_{L_{x,v}^2}^2 \\ (7.1) \quad &\quad + K(A, \alpha, b, a) (\mathcal{G}_{x,v}^1(h, h))^2, \end{aligned}$$

with K a fonction only depending on the coefficients appearing in hypocoercivity hypothesis and independent of ε .

We directly see that we have exactly the same kind of bound as the one we obtain while working on the a priori estimates for the operator $\|h\|_{\mathcal{H}_\varepsilon^1}$, equation (5.1). Therefore we can choose of coefficients A, α, b, e and a in the same way (in the right order) and use the same inequalities to finally obtain the expected result: $\exists K_0, K_1 > 0, \forall 0 < \varepsilon \leq 1$,

$$\begin{aligned} \frac{d}{dt} \|h\|_{\mathcal{H}_{\varepsilon^\perp}^1}^2 &\leq -K_0^{(1)} \left(\frac{1}{\varepsilon^2} \|h^\perp\|_\Lambda^2 + \frac{1}{\varepsilon^2} \|\nabla_x h^\perp\|_\Lambda^2 + \frac{1}{\varepsilon^2} \|\nabla_v h^\perp\|_\Lambda^2 + \|\nabla_x h\|_{L_{x,v}^2}^2 \right) \\ &\quad + K_1^{(1)} (\mathcal{G}_{x,v}^1(h, h))^2, \end{aligned}$$

with the constants $K_0^{(1)}$ and $K_1^{(1)}$ independent of ε , and $\|h\|_{\mathcal{H}_{1\perp}^1}^2$ equivalent to $\|h\|_{L_{x,v}^2}^2 + \|\nabla_x h\|_{L_{x,v}^2}^2 + \|\nabla_v h^\perp\|_{L_{x,v}^2}^2$. Therefore, for all ε small enough we have the expected result in the case $k = 1$.

7.1.2. The induction in higher order Sobolev spaces. Then we assume that the theorem is true up to the integer $k - 1$, $k > 1$. Then we suppose that L satisfies (H1'), (H2') and (H3) and we consider ε in $(0, 1]$.

Since h_{in} is in $\text{Ker}(G_\varepsilon)^\perp$, h belongs to $\text{Ker}(G_\varepsilon)^\perp$ for all t and so we can use the results given in the toolbox.

As in the proofs of previous sections, we define on $H_{x,v}^k$:

$$\begin{aligned} F_k(t) &= \sum_{\substack{|j|+|l|=k \\ |j|\geq 2}} B \|\partial_l^j h^\perp\|_{L_{x,v}^2}^2 + B' \sum_{\substack{|l|=k \\ i, c_i(l)>0}} Q_{l,i}(t), \\ Q_{l,i}(t) &= \alpha \|\partial_l^0 h\|_{L_{x,v}^2}^2 + b \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_{L_{x,v}^2}^2 + a\varepsilon \langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2}, \end{aligned}$$

where the constants, strictly positive, will be chosen later.

Like in the section above, we shall study the time evolution of every term involved in F_k in order to bound above $\frac{dF_k}{dt}(t)$ with expected coefficients. However, in this subsection we will need to control all the $Q_{l,i}$ in the same time rather than treated them separately as we did in the proof of Proposition (2.2), because the toolbox tells us that each $Q_{l,i}$ is controlled by quantities appearing in the others.

7.1.3. The time evolution of $\sum Q_{l,i}$. Gathering the toolbox inequalities in the following way: $\alpha(3.10) + b(3.17) + a\varepsilon(3.18)$. This yields, because $0 < \varepsilon \leq 1$ and $\text{Card}\{i, c_i(l) > 0\} \leq N$,

$$\begin{aligned} \frac{d}{dt} \left(\sum_{\substack{|l|=k \\ i, c_i(l)>0}} Q_{l,i}(t) \right) &\leq \frac{1}{\varepsilon^2} \left[\tilde{K}^\perp e a - \lambda \alpha \right] \sum_{|l|=k} \|\partial_l^0 h^\perp\|_\Lambda^2 \\ &\quad + \frac{1}{\varepsilon^2} \left[\frac{1}{4C_{\pi k} C_\pi N} \frac{a}{e} - \nu_5^\Lambda b \right] \sum_{\substack{|l|=k \\ i, c_i(l)>0}} \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_\Lambda^2 \\ &\quad + \left[K_{dl}^\perp N b - \frac{a}{2} \right] \sum_{|l|=k} \|\partial_l^0 h\|_{L_{x,v}^2}^2 + \frac{a}{4} \sum_{|l|\leq k-1} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \\ &\quad + \frac{bK_{k-1}^\perp}{\varepsilon^2} \left(\sum_{\substack{|l|+|j|=k \\ i, c_i(l)>0}} 1 \right) \|h^\perp\|_{H_{x,v}^{k-1}}^2 + K(\alpha, b, a, e) (\mathcal{G}_{x,v}^k(h, h))^2, \end{aligned}$$

with K a fonction only depending on the coefficients appearing in hypocoercivity hypothesis and independent of ε .

One can notice that except for the terms in $\|h\|_{H_{x,v}^{k-1}}$ and $\sum_{|l|\leq k-1} \|\partial_l^0 h\|_{L_{x,v}^2}^2$, we have exactly the same bound as in the case $k = 1$, equation (7.1). Therefore we can

choose α, b, a, e , independently of ε and Γ such that it exists $K'_0 > 0, K'_1 > 0$ and $C_0, C_1 > 0$ such that for all $0 < \varepsilon \leq 1$:

$$\begin{aligned}
& \bullet \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} Q_{l,i}(t) \sim \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} \left(\|\partial_l^0 h\|_{L_{x,v}^2}^2 + \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_{L_{x,v}^2}^2 \right), \\
& \bullet \frac{d}{dt} \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} Q_{l,i}(t) \leq -K'_0 \left(\frac{1}{\varepsilon^2} \sum_{|l|=k} \|\partial_l^0 h^\perp\|_{\Lambda}^2 + \frac{1}{\varepsilon^2} \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_{\Lambda}^2 \right. \\
& \quad \left. + \sum_{|l|=k} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \right) \\
& \quad + \frac{C_0}{\varepsilon^2} \|h^\perp\|_{H_{x,v}^{k-1}}^2 + C_1 \sum_{|l| \leq k-1} \|\partial_l^0 h\|_{L_{x,v}^2}^2 + K'_1 (\mathcal{G}_{x,v}^k(h, h))^2.
\end{aligned}$$

7.1.4. *The time evolution of F_k and conclusion.* We can finally obtain the time evolution of F_k , using $\frac{d}{dt} \|\partial_l^j h^\perp\|_{L_{x,v}^2}^2$, equation (3.16), so that there is no more ε in front of the Γ term.

$$\begin{aligned}
\frac{d}{dt} F_k(t) & \leq -B \frac{\nu_5^\Lambda}{\varepsilon^2} \sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} \|\partial_l^j h^\perp\|_{\Lambda}^2 + B \frac{9(\nu_1^\Lambda)^2 N}{2(\nu_0^\Lambda)^2 \nu_5^\Lambda \varepsilon^2} \sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} \sum_{i, c_i(j) > 0} \|\partial_{l+\delta_i}^{j-\delta_i} h\|_{\Lambda}^2 \\
& - K'_0 B' \left(\frac{1}{\varepsilon^2} \sum_{|l|=k} \|\partial_l^0 h^\perp\|_{\Lambda}^2 + \frac{1}{\varepsilon^2} \sum_{\substack{|l|=k \\ i, c_i(l) > 0}} \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_{\Lambda}^2 + \sum_{|l|=k} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \right) \\
& + \left(\sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} B K_{dl}^\perp + B' C_1 \right) \sum_{|l| \leq k-1} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \\
& + \frac{1}{\varepsilon^2} \left[\sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} B K_{k-1}^\perp + B' C_0 \right] \|h^\perp\|_{H_{x,v}^{k-1}}^2 \\
& + \left[\sum_{\substack{|j|+|l|=k \\ |j| \geq 2}} \frac{3B\nu_1^\Lambda}{\nu_0^\Lambda \nu_5^\Lambda} + B' K'_1 \right] (\mathcal{G}_{x,v}^k(h, h))^2,
\end{aligned}$$

Therefore we obtain the same bound (except $\sum_{|l| \leq k-1} \|\partial_l^0 h\|_{L_{x,v}^2}^2$) as in the proof of Proposition 2.2, equation (5.2), and so by choosing coefficients in the same way we have that it exists $C_+^{(k)} > 0, 0 < \varepsilon_N \leq 1$ and $K_1^{(k*)} > 0$, none of them depending on ε , such that for all $0 < \varepsilon \leq \varepsilon_N$:

$$\begin{aligned}
\frac{d}{dt} F_k(t) \leq & C_+^{(k)} \left(\frac{1}{\varepsilon^2} \sum_{|j|+|l| \leq k-1} \|\partial_l^j h^\perp\|_\Lambda^2 + \sum_{|l| \leq k-1} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \right) \\
& - \left(\frac{1}{\varepsilon^2} \sum_{|j|+|l|=k} \|\partial_l^j h^\perp\|_\Lambda^2 + \sum_{|l|=k} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \right) \\
& + K_1^{(k*)} (\mathcal{G}_{x,v}^k(h, h))^2.
\end{aligned}$$

This inequality is true for all k and therefore we can take a linear combination of the F_k to obtain the required result. Using the induction hypothesis on F_1 up to F_{k-1} we also have the equivalence of norms.

7.2. The exponential decay: proof of Theorem 2.4. Thanks to Theorem 2.3, we know that we have solution to the equation (1.3) for given h_{in} small enough in the standard Sobolev norm. Call h the associated solution of $h_{in} \in H_{x,v}^k$ to (1.3). Since the existence has been proved we can use the a priori estimate above and the Proposition 7.1.

Thus we have

$$\frac{d}{dt} \|h\|_{\mathcal{H}_{\varepsilon^\perp}^k}^2 \leq -K_0^{(k)} \left(\frac{1}{\varepsilon^2} \|h^\perp\|_{H_\Lambda^k}^2 + \sum_{1 \leq |l| \leq k} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \right) + K_1^{(k)} (\mathcal{G}_{x,v}^k(h, h))^2.$$

As before we can use (3.4) (equivalence of norms $L_{x,v}^2$ and Λ on the fluid part) to get, for $|l| > 1$,

$$\|\partial_l^0 h\|_\Lambda^2 \leq C' \left(\|\partial_l^0 h^\perp\|_\Lambda^2 + \|\partial_l^0 h\|_{L_{x,v}^2}^2 \right),$$

and for the case $|l| \leq 1$ we can apply the Poincare inequality (3.5) together with the equivalence of the $L_{x,v}^2$ -norm and the Λ -norm on the fluid part π_L , (3.4) to get

$$\exists C, C' > 0, \begin{cases} \|h\|_\Lambda^2 & \leq C \left(\|h^\perp\|_\Lambda^2 + \frac{1}{2} \|\nabla_x h\|_{L_{x,v}^2}^2 \right), \\ \|\nabla_x h\|_\Lambda^2 & \leq C' \left(\|\nabla_x h^\perp\|_\Lambda^2 + \frac{1}{2} \|\nabla_x h\|_{L_{x,v}^2}^2 \right). \end{cases}$$

Then we get that

$$\begin{aligned}
\frac{d}{dt} \|h\|_{\mathcal{H}_{\varepsilon^\perp}^k}^2 & \leq -K_0^{(k)} \left(\sum_{\substack{|j|+|l| \leq k \\ |j| \geq 1}} \|\partial_l^j h^\perp\|_\Lambda^2 + \sum_{|l| \leq k} \|\partial_l^0 h\|_\Lambda^2 \right) + K_1^{(k)} (\mathcal{G}_{x,v}^k(h, h))^2 \\
& \leq -K_0^{(k*)} \|h\|_{H_\Lambda^k}^2 + K_1^{(k)} (\mathcal{G}_{x,v}^k(h, h))^2.
\end{aligned}$$

Then for $k \geq k_0$, defined in (H4), and because Γ satisfies (H4) we can write

$$\frac{d}{dt} \|h\|_{\mathcal{H}_{\varepsilon^\perp}^k}^2 \leq \left(K_1^{(k)} C_\Gamma^2 \|h\|_{H_{x,v}^k}^2 - K_0^{(k*)} \right) \|h\|_{H_\Lambda^k}^2.$$

Because $\|h\|_{\mathcal{H}_{\varepsilon\perp}^k}$ and $\|h\|_{H_{x,v}^k}^2$ are equivalent, independently of ε , we finally have

$$\frac{d}{dt} \|h\|_{\mathcal{H}_{\varepsilon\perp}^k}^2 \leq \left(K_1^{(k)} C_F^2 C \|h\|_{\mathcal{H}_{\varepsilon\perp}^k}^2 - K_0^{(k*)} \right) \|h\|_{H_\Lambda^k}^2.$$

Therefore if

$$\|h_{in}\|_{\mathcal{H}_{\varepsilon\perp}^k}^2 \leq \frac{K_0^{(k*)}}{2K_1^{(k)} C_F^2 C}$$

we have that $\|h\|_{\mathcal{H}_{\varepsilon\perp}^k}^2$ is always decreasing on \mathbb{R}^+ and so for all $t > 0$

$$\frac{d}{dt} \|h\|_{\mathcal{H}_{\varepsilon\perp}^k}^2 \leq -\frac{K_0^{(k*)}}{2K_1^{(k)} C_F^2 C} \|h\|_{H_\Lambda^k}^2.$$

And the H_Λ^k -norm controls the $H_{x,v}^k$ -norm which is equivalent to the $\mathcal{H}_{\varepsilon\perp}^k$ -norm. Thus applying Gronwall's lemma gives us the expected exponential decay.

8. INCOMPRESSIBLE NAVIER-STOKES LIMIT: PROOF OF THEOREM 2.5

In this section we consider $k \geq k_0$, $0 < \varepsilon \leq \varepsilon_N$ and we take h_{in} in $H_{x,v}^k$ such that $\|h_{in}\|_{\mathcal{H}_{\varepsilon}^k} \leq \delta_k$.

Therefore we know, thanks to theorem 2.3, that we have a solution h_ε to the linearized Boltzmann equation

$$\partial_t h_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x h_\varepsilon = \frac{1}{\varepsilon^2} L(h_\varepsilon) + \frac{1}{\varepsilon} \Gamma(h_\varepsilon, h_\varepsilon),$$

with $h_\varepsilon(0, x, v) = h_{in}(x, v)$. Moreover, we also know that (h_ε) tends weakly* to h in $L_t^\infty(H_x^k L_v^2)$.

We proved in the linear case, theorem 2.1, that the linear operator $G_\varepsilon = \varepsilon^{-2} L - \varepsilon^{-1} v \cdot \nabla_x$ generates a semigroup e^{tG_ε} on $H_{x,v}^k$. Therefore we can use Duhamel's principle to rewrite our equation under the following form, defining $u_\varepsilon = \Gamma(h_\varepsilon, h_\varepsilon)$,

$$\begin{aligned} h_\varepsilon &= e^{tG_\varepsilon} h_{in} + \int_0^t \frac{1}{\varepsilon} e^{(t-s)G_\varepsilon} u_\varepsilon(s) ds \\ (8.1) \quad &:= U^\varepsilon h_{in} + \Psi^\varepsilon(u_\varepsilon). \end{aligned}$$

The article by Ellis and Pinsky [9] gives us a Fourier theory in x of the semigroup e^{tG_ε} and therefore we are going to use it to study the strong limit of $U^\varepsilon h_{in}$ and $\Psi^\varepsilon(u_\varepsilon)$ as ε tends to 0. We will denote by \mathcal{F}_x the Fourier transform in x on the torus (which is discrete) and n the discrete variable associated in \mathbb{Z}^N .

From [9], we are using Theorem 3.1, rewritten thanks with the Proposition 2.6 and the Appendix II with $\delta = \lambda/4$ in Proposition 2.3, to get the following theorem

Theorem 8.1. *There exists $n_0 \in \mathbb{R}^{*+}$, there exists functions*

- $\lambda_j : [-n_0, n_0] \longrightarrow \mathbb{C}$, $-1 \leq j \leq 2$, C^∞
- $e_j : [-n_0, n_0] \times \mathbb{S}^{N-1} \longrightarrow L_v^2$, $-1 \leq j \leq N$, C^∞ in ζ and C^0 in ω ,
 $(\zeta, \omega) \longmapsto e_j(\zeta, \omega)$

such that

- (1) for all $-1 \leq j \leq 2$, $\lambda_j(\zeta) = i\alpha_j \zeta - \beta_j \zeta^2 + \gamma_j(\zeta)$, where $\alpha_j \in \mathbb{R}$, with $\alpha_0 = \alpha_2 = 0$, $\beta_j < 0$ and $|\gamma_j(\zeta)| \leq C_\gamma |\zeta|^3$ with $|\gamma_j(\zeta)| \leq \frac{\beta_j}{2} |\zeta|^2$,

- (2) for all $-1 \leq j \leq N$
- $e_j(\zeta, \omega) = e_{0j}(\omega) + \zeta e_{1j}(\omega) + \zeta^2 e_{2j}(\zeta, \omega),$
 - $e_{0-1}(\omega)(v) = e_{01}(-\omega)(v) = A \left(1 - \omega \cdot v + \frac{|v|^2 - N}{2} \right) M(v)^{1/2},$
- (3) we have $e^{tG_\varepsilon} = \mathcal{F}_x^{-1} \hat{U}(t/\varepsilon^2, \varepsilon n, v) \mathcal{F}_x$ where

$$\hat{U}(t, n, v) = \sum_{j=-1}^2 \hat{U}_j(t, n, v) + \hat{U}_R(t, n, v)$$

with the following properties

- for $-1 \leq j \leq 2, \hat{U}_j(t, n, v) = \chi_{|n| \leq n_0} e^{t\lambda_j(|n|)} P_j \left(|n|, \frac{n}{|n|} \right) (v),$
- for $-1 \leq j \leq 1, P_j \left(|n|, \frac{n}{|n|} \right) = e_j \left(|n|, \frac{n}{|n|} \right) \otimes e_j \left(|n|, \frac{-n}{|n|} \right),$
- $P_2 \left(|n|, \frac{n}{|n|} \right) = \sum_{j=2}^N e_j \left(|n|, \frac{n}{|n|} \right) \otimes e_j \left(|n|, \frac{-n}{|n|} \right),$
- for $-1 \leq j \leq 2, P_j \left(|n|, \frac{n}{|n|} \right) = P_{0j} \left(\frac{n}{|n|} \right) + |n| P_{1j} \left(\frac{n}{|n|} \right) + |n|^2 P_{2j} \left(|n|, \frac{n}{|n|} \right),$
- $\sum_{j=-1}^2 P_{0j} = \pi_L,$
- it exists $C_R, \sigma > 0$ such that for all $t \in \mathbb{R}^+$ and all $n \in \mathbb{Z}^N,$

$$|||\hat{U}_R(t, n, v)|||_{L_v^2} \leq C_R e^{-\sigma t}.$$

This theorem gives us all the tools we need to study the convergence as ε tends to 0 since we have an explicit form for the Fourier transform of the semigroup. We also know that this semigroup commutes with the pure x -derivatives. Therefore, studying the convergence in the $L_x^2 L_v^2$ -norm will be enough to obtain the desired result in the $H_x^k L_v^2$ -norm.

We are going to prove the following convergences in the different settings stated by Theorem 2.5

- (1) $U^\varepsilon h_{in}$ tends to $V(t, x, v) h_{in}$ with $V(0)$ the projection on the subset of $\text{Ker}(L)$ consisting in functions g such that $\nabla_x \cdot u_g = 0$ and $\rho_g + \theta_g = 0,$
- (2) $\Psi^\varepsilon(u_\varepsilon)$ converges to $\Psi(h, h)$ with $\Psi(h, h)(t=0) = 0.$

8.1. Study of the linear part. We remind here that we have

$$U^\varepsilon h_{in} = \mathcal{F}_x^{-1} \hat{U}^\varepsilon(t, n, v) \hat{h}_{in}(n, v)$$

with

$$\begin{aligned} \hat{U}^\varepsilon(t, n, v) &= \sum_{j=-1}^2 \hat{U}_j^\varepsilon(t, n, v) + \hat{U}_R^\varepsilon(t, n, v), \\ \hat{U}_j^\varepsilon(t, n, v) &= \chi_{|\varepsilon n| \leq n_0} e^{\frac{i\alpha_j t |n|}{\varepsilon} - \beta_j t |n|^2 + \frac{t}{\varepsilon^2} \gamma_j(|\varepsilon n|)} \left[P_{0j} \left(\frac{n}{|n|} \right) + \varepsilon |n| \tilde{P}_{1j} \left(|\varepsilon n|, \frac{n}{|n|} \right) \right] \end{aligned}$$

We can decompose \hat{U}_j^ε into four different terms

$$\begin{aligned}
(8.2) \quad \hat{U}_j^\varepsilon(t, n, v) &= e^{\frac{i\alpha_j t|n|}{\varepsilon} - \beta_j t|n|^2} P_{0j} \left(\frac{n}{|n|} \right) \\
&+ \chi_{|\varepsilon n| \leq n_0} e^{\frac{i\alpha_j t|n|}{\varepsilon} - \beta_j t|n|^2} \left(e^{\frac{t}{\varepsilon^2} \gamma_j(|\varepsilon n|)} - 1 \right) P_{0j} \left(\frac{n}{|n|} \right) \\
&+ \chi_{|\varepsilon n| \leq n_0} e^{\frac{i\alpha_j t|n|}{\varepsilon} - \beta_j t|n|^2 + \frac{t}{\varepsilon^2} \gamma_j(|\varepsilon n|)} \varepsilon |n| \tilde{P}_{1j} \left(|\varepsilon n|, \frac{n}{|n|} \right) \\
&+ (\chi_{|\varepsilon n| \leq n_0} - 1) e^{\frac{i\alpha_j t|n|}{\varepsilon} - \beta_j t|n|^2} P_{0j} \left(\frac{n}{|n|} \right). \\
&:= U_{0j}^\varepsilon + U_{1j}^\varepsilon + U_{2j}^\varepsilon + U_{3j}^\varepsilon.
\end{aligned}$$

Remark 8.2. One can notice that U_{00}^ε and U_{02}^ε do not depend on ε , since $\alpha_0 = \alpha_2 = 0$.

We are going to study each of these four terms in two different lemmas and then add a last lemma to deal with the remainder term $U_R h_{in}$. The Lemmas will be proven in Appendix 3.

Lemma 8.3. For $\alpha_j \neq 0$ ($j = \pm 1$) we have that it exists $C_0 > 0$ such that for all $T \in [0, +\infty]$

$$\left\| \int_0^T U_{0j}^\varepsilon h_{in} dt \right\|_{L_x^2 L_v^2}^2 \leq C_0 \varepsilon^2 \|h_{in}\|_{L_x^2 L_v^2}^2.$$

Moreover we have a strong convergence in the $L_{[0, +\infty)}^2 L_x^2 L_v^2$ -norm if and only if h_{in} satisfies $\nabla_x \cdot u_{in} = 0$ and $\rho_{in} + \theta_{in} = 0$. In that case we have $U_{0j}^\varepsilon h_{in} = 0$.

Lemma 8.4. For $-1 \leq j \leq 2$ and for $1 \leq l \leq 3$ we have that the three following inequalities hold for U_{lj}^ε

- $\exists C_l > 0, \forall T > 0, \quad \left\| \int_0^T U_{lj}^\varepsilon h_{in} dt \right\|_{L_x^2 L_v^2}^2 \leq C_l \varepsilon^2 \|h_{in}\|_{L_x^2 L_v^2}^2,$
- $\exists C'_l > 0, \quad \|U_{lj}^\varepsilon h_{in}\|_{L_{[0, +\infty)}^2 L_x^2 L_v^2}^2 \leq C'_l \varepsilon^2 \|h_{in}\|_{L_x^2 L_v^2}^2,$
- $\forall \delta \in [0, 1], \exists C_\delta^{(l)} > 0, \forall t > 0, \quad \|U_{lj}^\varepsilon h_{in}(t)\|_{L_x^2 L_v^2}^2 \leq C_\delta^{(l)} \varepsilon^{2\delta} \|h_{in}\|_{H_x^\delta L_v^2}^2.$

Lemma 8.5. For the remainder term we have the two following inequalities

- $\exists C_4 > 0, \forall T > 0, \quad \left\| \int_0^T U_R^\varepsilon h_{in} dt \right\|_{L_x^2 L_v^2}^2 \leq C_4 T \varepsilon^2 \|h_{in}\|_{L_x^2 L_v^2}^2,$
- $\exists C'_4 > 0, \quad \|U_R^\varepsilon h_{in}\|_{L_{[0, +\infty)}^2 L_x^2 L_v^2}^2 \leq C'_4 \varepsilon^2 \|h_{in}\|_{L_x^2 L_v^2}^2,$
- $\forall t_0 > 0, \exists C_r > 0, \forall t > t_0, \quad \|U_R h_{in}(t)\|_{L_x^2 L_v^2}^2 \leq \frac{C_r}{\sqrt{t_0}} \varepsilon \|h_{in}\|_{L_x^2 L_v^2}^2.$

Moreover, the strong convergence up to $t_0 = 0$ is possible if and only if h_{in} is in $\text{Ker}(L)$. In that case we have

$$\forall \delta \in [0, 1], \exists C_\delta^{(R)} > 0, \forall t > 0, \|U_R^\varepsilon h_{in}\|_{L_x^2 L_v^2}^2 \leq C_\delta^{(R)} \varepsilon^{2\delta} \|h_{in}\|_{H_x^\delta L_v^2}^2.$$

Therefore, gathering lemmas 8.3, 8.4 and 8.5 and reminding Remark 8.2, we proved that, as ε tends to 0, $(e^{tG_\varepsilon} h_{in})$ converges to

$$(8.3) \quad V(t, x, v)h_{in}(x, v) = \mathcal{F}_x^{-1} \left[e^{-\beta_0 t |n|^2} P_{00} \left(\frac{n}{|n|} \right) + e^{-\beta_2 t |n|^2} P_{02} \left(\frac{n}{|n|} \right) \right] \mathcal{F}_x h_{in}.$$

The convergence is strong when we consider the average in time and is strong in $L_t^2 H_x^k L_v^2$ (and in $C([0, +\infty), H_x^k L_v^2)$ if h_{in} is in $H_x^{k+0} L_v^2$) if and only if both conditions found in Lemma 8.3 and Lemma 8.5 are satisfied. That is to say h_{in} belongs to $\text{Ker}(L)$ with $\nabla_x \cdot u_{in} = 0$ and $\rho_{in} + \theta_{in} = 0$.

Moreover this also allows us to see that $V(0)$ is the projection on the subset of $\text{Ker}(L)$ consisting in functions g such that $\nabla_x \cdot u_g = 0$ and $\rho_g + \theta_g = 0$.

8.2. Study of the bilinear part. We recall here that $u_\varepsilon = \Gamma(h_\varepsilon, h_\varepsilon)$. Therefore, by hypothesis (H5), u_ε belongs to $\text{Ker}(L)^\perp$. Then we know that for all $-1 \leq j \leq 2$, $P_{0j} \left(\frac{n}{|n|} \right)$ is a projection onto a subspace of $\text{Ker}(L)$. Therefore we have that, in the Fourier space,

$$P_j \left(|\varepsilon n|, \frac{n}{|n|} \right) \hat{u}_\varepsilon = |\varepsilon n| P_{1j} \left(\frac{n}{|n|} \right) \hat{u}_\varepsilon + |\varepsilon n|^2 P_{2j} \left(|\varepsilon n|, \frac{n}{|n|} \right) \hat{u}_\varepsilon.$$

Thus, recalling that

$$\Psi^\varepsilon(u_\varepsilon) = \int_0^t \frac{1}{\varepsilon} e^{(t-s)G_\varepsilon} u_\varepsilon(s) ds,$$

we can decompose it

$$\Psi^\varepsilon(u_\varepsilon) = \sum_{j=-1}^2 \psi_j^\varepsilon(u_\varepsilon) + \psi_R^\varepsilon(u_\varepsilon),$$

with

$$\begin{aligned} \psi_j^\varepsilon(u_\varepsilon) &= \mathcal{F}_x^{-1} \chi_{|\varepsilon n| \leq n_0} \int_0^t e^{\frac{i\alpha_j(t-s)|n|}{\varepsilon} - \beta_j(t-s)|n|^2 + \frac{t-s}{\varepsilon^2} \gamma_j(|\varepsilon n|)} |n| (P_{1j} + \varepsilon |n| P_{2j}) \hat{u}_\varepsilon(s) ds. \\ &:= \psi_{0j}^\varepsilon(u_\varepsilon) + \psi_{1j}^\varepsilon(u_\varepsilon) + \psi_{2j}^\varepsilon(u_\varepsilon) + \psi_{3j}^\varepsilon(u_\varepsilon), \end{aligned}$$

where we have used the same decomposition as in the linear case, equation (8.2), substituting t by $t - s$, P_{0j} by $|n| P_{1j}$ and \tilde{P}_{1j} by $|n| P_{2j}$. And

$$\psi_R^\varepsilon(u_\varepsilon) = \int_0^t \frac{1}{\varepsilon} U_R^\varepsilon(t-s) u_\varepsilon(s) ds.$$

Like the linear case, Remark 8.2, ψ_{00}^ε and ψ_{02}^ε do not depend on ε and we are going to prove the convergence towards $\Psi(u) = \mathcal{F}_x^{-1} [\psi_{00}^\varepsilon(u) + \psi_{02}^\varepsilon(u)] \mathcal{F}_x$, where $u = \Gamma(h, h)$. To establish such a result we are going to study each term in three different lemmas and then a fourth one will deal with the remainder term. The Lemmas will be proven in Appendix 3.

Lemma 8.6. *For $\alpha_j \neq 0$ ($j = \pm 1$) we have the following inequality for ψ_{0j}^ε :*

$$\exists \tilde{C}_0 > 0, \forall T > 0, \quad \left\| \int_0^T \psi_{0j}^\varepsilon(u_\varepsilon) dt \right\|_{L_x^2 L_v^2}^2 \leq \tilde{C}_0 T^2 \varepsilon^2 \sup_{t \in [0, T]} \|h_\varepsilon(t, x, v)\|_{L_x^2 L_v^2}^4.$$

Remark 8.7. We know that $(h_\varepsilon)_{\varepsilon>0}$ is bounded in $L_t^\infty H_x^k L_v^2$ (see theorems 2.3 and 2.4).

This remark gives us the strong convergence to 0 of the average in time and the strong convergence to 0 without averaging in time as long as h_{in} belongs to $\text{Ker}(L)$ in Lemma 8.6.

Lemma 8.8. For $-1 \leq j \leq 2$ and for $1 \leq l \leq 3$ we have that the three following inequalities hold for ψ_{lj}^ε

- $\exists \tilde{C}_l > 0, \forall T > 0, \quad \left\| \int_0^T \psi_{lj}^\varepsilon(u_\varepsilon) dt \right\|_{L_x^2 L_v^2}^2 \leq \tilde{C}_l T^{5/2} \varepsilon \sup_{t \in [0, T]} \|h_\varepsilon(t, x, v)\|_{L_x^2 L_v^2}^4,$
- $\exists \tilde{C}_l' > 0, \forall T > 0, \quad \|\psi_{lj}^\varepsilon(u_\varepsilon)\|_{L_{[0, T]}^2 L_x^2 L_v^2}^2 \leq \tilde{C}_l' T^{3/2} \varepsilon \sup_{t \in [0, T]} \|h_\varepsilon(t, x, v)\|_{L_x^2 L_v^2}^4,$
- $\exists \tilde{C}_l'' > 0, \forall T > 0, \quad \|\psi_{lj}^\varepsilon(u_\varepsilon)(T)\|_{L_x^2 L_v^2}^2 \leq \tilde{C}_l'' T^{1/2} \varepsilon \sup_{t \in [0, T]} \|h_\varepsilon(t, x, v)\|_{L_x^2 L_v^2}^4.$

Lemma 8.9. For the remainder term we have the three following inequalities

- $\exists \tilde{C}_4 > 0, \forall T > 0, \quad \left\| \int_0^T \psi_R^\varepsilon(u_\varepsilon) dt \right\|_{L_x^2 L_v^2}^2 \leq \tilde{C}_4 T^3 \varepsilon \sup_{t \in [0, T]} \|h_\varepsilon(t, x, v)\|_{L_x^2 L_v^2}^4,$
- $\exists \tilde{C}_4' > 0, \forall T > 0, \quad \|\psi_R^\varepsilon(u_\varepsilon)\|_{L_{[0, T]}^2 L_x^2 L_v^2}^2 \leq \tilde{C}_4' T^2 \varepsilon \sup_{t \in [0, T]} \|h_\varepsilon(t, x, v)\|_{L_x^2 L_v^2}^4,$
- $\exists \tilde{C}_4'' > 0, \forall T > 0, \quad \|\psi_R^\varepsilon(u_\varepsilon)(T)\|_{L_x^2 L_v^2}^2 \leq \tilde{C}_4'' T \varepsilon \sup_{t \in [0, T]} \|h_\varepsilon(t, x, v)\|_{L_x^2 L_v^2}^4.$

Gathering all Lemmas 8.6, 8.8 and 8.9 gives us the strong convergence of $\Psi^\varepsilon(u_\varepsilon) - \Psi(u_\varepsilon)$ towards 0, thanks to Remark 8.7. It remains to prove that we have indeed the expected convergences of $\Psi(u_\varepsilon)$ towards $\Psi(u)$ as ε tends to 0.

We start this last step by a quick remark relying on Sobolev embeddings and giving us a strong convergence of h_ε towards h in $L_{[0, T]}^\infty L_x^\infty L_v^2$, for $T > 0$.

Remark 8.10. We know that $h_\varepsilon \rightarrow h$ weakly- $*$ in $L_t^\infty H_x^k L_v^2$, for $k \geq k_0 > N/2$. But we also proved that for all $t > 0$ that $(h_\varepsilon)_\varepsilon$ is bounded in $H_x^k L_v^2$. Therefore the sequence $(\|h_\varepsilon\|_{L_v^2}, \varepsilon > 0)$ is bounded in H_x^k and therefore convergence strongly in $H_x^{k'}$ for all $k' < k$.

But, by triangular inequality it comes that

$$\left| \|h_\varepsilon\|_{H_x^{k'} L_v^2} - \|h\|_{H_x^{k'} L_v^2} \right| \leq \left\| \|h_\varepsilon\|_{L_v^2} - \|h\|_{L_v^2} \right\|_{H_x^{k'}}.$$

This means that we also have that $\lim_{\varepsilon \rightarrow 0} \|h_\varepsilon\|_{H_x^{k'} L_v^2} = \|h\|_{H_x^{k'} L_v^2}$. The space $H_x^{k'} L_v^2$ is a Hilbert space and h_ε tends weakly to h in it, therefore the last result gives us that in fact h_ε tends strongly to h in $H_x^{k'} L_v^2$.

This result is for all $t > 0$ and all $k' \leq k$. Furthermore, $k > N/2$ and so we can choose $k' > N/2$. By Sobolev's embedding we obtain that h_ε tends strongly to h in $L_x^\infty L_v^2$, for all $t > 0$. Reminding that $h_\varepsilon \rightarrow h$ weakly- $*$ in $L_t^\infty H_x^k L_v^2$ and we obtain that we have

$$\forall T > 0, \quad V_T(\varepsilon) = \sup_{t \in [0, T]} \|h_\varepsilon - h\|_{L_x^\infty L_v^2} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

Lemma 8.11. We have the following rate of convergence:

- $\exists \tilde{C}_5 > 0, \forall T > 0, \left\| \int_0^T \Psi(u_\varepsilon) dt - \int_0^T \Psi(u) dt \right\|_{L_x^2 L_v^2}^2 \leq \tilde{C}_5 T^2 V_T(\varepsilon)^2,$
- $\exists \tilde{C}'_5 > 0, \forall T > 0, \|\Psi(u_\varepsilon) - \Psi(u)\|_{L_{[0,T]}^2 L_x^2 L_v^2}^2 \leq \tilde{C}'_5 T V_T(\varepsilon)^2,$
- $\exists \tilde{C}''_5 > 0, \forall T > 0, \|\Psi(u_\varepsilon) - \Psi(u)\|_{L_x^2 L_v^2}^2(T) \leq \tilde{C}''_5 V_T(\varepsilon)^2.$

Thus, those Lemmas, combined with the study of the linear case (Lemmas 8.3, 8.4 and 8.5) prove the Theorem 2.5 with the rate of convergence being the maximum of each rate of convergence. Moreover we have proved

$$h(t, x, v) = V(t, x, v) h_{in}(x, v) + \Psi(t, x, v)(\Gamma(h, h)).$$

APPENDIX A. APPENDIX 1: VALIDATION OF THE ASSUMPTIONS

As said in the introduction, all the hypocoercivity theory assumptions hold for several different kinetic models. One can find the proof of the assumptions (H1), (H2), (H3), (H1') and (H2') in [20] directly for the linear relaxation (see also [5]), the semi-classical relaxation (see also [22]), the linear Fokker-Planck equation, the Boltzmann equation with hard potential and angular cutoff and the Landau equation with hard and moderately soft potential (both studied in a constructive way in [18] and [2], for the spectral gaps, see also [13] and [14] for the Cauchy problems):

- The Linear Relaxation

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \left[\left(\int_{\mathbb{R}^N} f(t, x, v_*) dv_* \right) M(v) - f \right],$$

- The Semi-classical Relaxation

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \int_{\mathbb{R}^N} [M(1 - \delta f) f_* - M_*(1 - \delta f_*) f] dv_*,$$

- The Linear Fokker-Planck Equation

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \nabla_{v \cdot} (\nabla_v f + f v),$$

- The Boltzmann Equation with hard potential and angular cutoff

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} b(\cos \theta) |v - v_*|^\gamma [f' f'_* - f f_*] dv_* d\sigma,$$

- The Landau Equation with hard and moderately soft potential

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \nabla_{v \cdot} \left(\int_{\mathbb{R}^N} \Phi(v - v_*) |v - v_*|^{\gamma+2} [f_*(\nabla f) - f(\nabla f)_*] \right).$$

Assumption (H4) is clearly satisfied by the first three as in that case we have either $\|\cdot\|_{\Lambda_v} = \|\cdot\|_{L_v^2}$ or $\Gamma = 0$ (see [20]). Moreover, (H5) is obvious in the case of a linear equation. It thus remains to prove properties (H5) for the semi-classical relaxation and (H4) and (H5) for the Boltzmann equation and the Landau equation (since our property (H4) is slightly different from (H4) in [20]).

A.1. The semi-classical relaxation. In the case of the semi-classical relaxation, the linearization is slightly different. Indeed, the unique global equilibrium associated to an initial data f_0 is (assuming some initial bounds, see [20])

$$f_\infty = \frac{k_\infty M}{1 + \delta \kappa_\infty M},$$

where κ_∞ depends on f_0 .

Thus, we are no longer in the case of a global equilibrium being a Maxwellian. However, a good way of linearizing this equation is (see [20]) considering

$$f = f_\infty + \varepsilon \frac{\sqrt{\kappa_\infty M}}{1 + \delta \kappa_\infty M} h.$$

Using such a linearization instead of the one used all along this paper yields the same general equation (1.3) with L and Γ satisfying all the requirements (see [20]). Indeed, one may find that $\text{Ker}(L) = \text{Span}\left(f_\infty/\sqrt{M}\right)$ and then notice that this is not of the form needed in assumption (H3). However, this is bounded by $e^{-|v|^2/4}$ and therefore we are still able to use the toolbox (section 3, thus all the theorems).

Let us look at the bilinear operator to show that it fulfils hypothesis (H5). A straightforward computation gives us the definition of Γ ,

$$\Gamma(g, h) = \frac{\delta \sqrt{\kappa_\infty}}{2} \int_{\mathbb{R}^N} \sqrt{M_*} \frac{M_* - M}{1 + \varepsilon \kappa_\infty M_*} [hg_* + h_*g] dv_*.$$

Then, multiplying by a function f , integrating over \mathbb{R}^N and looking at the change of variable $(v, v_*) \rightarrow (v_*, v)$ yields

$$\langle \Gamma(g, h), f \rangle_{L_v^2} = \frac{\delta \sqrt{\kappa_\infty}}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N} (M_* - M)(gh_* + g_*h) \left[f \frac{\sqrt{M_*}}{1 + \delta \kappa_\infty M_*} - f_* \frac{\sqrt{M}}{1 + \delta \kappa_\infty M} \right] dv dv_*.$$

Therefore, taking f in $\text{Ker}(L)$ gives us the expected property.

A.2. Boltzmann operator with angular cutoff and hard potential. Notice that, compared to [20], we defined Γ in a way that it is symmetric which gives us, using the fact that $M_*M = M'_*M'$,

$$\Gamma(g, h) = \frac{1}{2} \int_{\mathbb{R}^N \times (S)^{N-1}} B(M^{1/2})_* [g'_*h' + g'h'_* - g_*h - gh_*] dv_* d\sigma,$$

A.2.1. Orthogonality to $\text{Ker}(L)$: (H5). A well-known property (see [10] for instance) tells us that for all ϕ in L_v^2 decreasing fast enough at infinity and for all ψ in L_v^2 one has

$$\begin{aligned} \int_{\mathbb{R}^N} \Gamma(g, h)(v) \psi(v) dv &= \frac{1}{8} \int_{(\mathbb{R}^N)^2 \times \mathbb{S}^{N-1}} B[g'_*h' + g'h'_* - g_*h - gh_*] \\ &\quad ((M^{1/2})'_*\psi + (M^{1/2})\psi_* - (M^{1/2})'_*\psi' - (M^{1/2})'\psi'_*) dv dv_* d\sigma. \end{aligned}$$

As shown in [20] or [7] we have that $\text{Ker}(L) = \text{Span}(1, v_1, \dots, v_N, |v|^2)M^{1/2}$ and therefore taking ψ to be each of these kernel functions gives us (H5).

A.2.2. *Controlling derivatives: (H4).* To prove (H4) we can define

$$\begin{aligned}\Gamma^+(g, h) &= \int_{\mathbb{R}^N \times (S)^{N-1}} B(M^{1/2})_* g'_* h' dv_* d\sigma, \\ \Gamma^-(g, h) &= - \int_{\mathbb{R}^N \times (S)^{N-1}} B(M^{1/2})_* g_* h dv_* d\sigma.\end{aligned}$$

By using the change of variable $u = v - v_*$ we end up with θ being a function of u and σ and $v' = v + f_1(u, \sigma)$ and $v'_* = v + f_2(u, \sigma)$, f_1 and f_2 being functions. Therefore we can make this change of variable, take j and l such that $|j| + |l| \leq k$ and differentiate our operator Γ^- .

$$\partial_l^j \Gamma^-(g, h) = -\frac{1}{2} \sum_{\substack{j_0+j_1+j_2=j \\ l_1+l_2=l}} \int_{\mathbb{R}^N \times S^{N-1}} b(\cos\theta) |u|^\gamma \partial_0^{j_0} (M(v-u)^{1/2}) \partial_{l_1}^{j_1} g_* \partial_{l_2}^{j_2} h dud\sigma.$$

Then we can easily compute that, C being a generic constant,

$$|\partial_0^{j_0} (M(v-u)^{1/2})| \leq CM(v-u)^{1/4}.$$

Moreover, we are in the case where $\gamma > 0$ and therefore we have

$$|u|^\gamma M(v-u)^{1/4} \leq C(1+|v|)^\gamma M(v-u)^{1/8}.$$

Combining this and the fact that $|b| \leq C_b$ (angular cutoff considered here), multiplying by a function f and integrating over $\mathbb{T}^N \times \mathbb{R}^N$ yields, using Cauchy-Schwarz two times,

$$\begin{aligned}|\langle \partial_l^j \Gamma^-(g, h), f \rangle_{L_{x,v}^2}| &\leq C \sum_{\substack{j_0+j_1+j_2=j \\ l_1+l_2=l}} \int_{\mathbb{T}^N \times \mathbb{R}^N} (1+|v|)^\gamma |\partial_{l_2}^{j_2} h| |f| \left(\int_{\mathbb{R}^N} M_*^{1/8} |\partial_{l_1}^{j_1} g_*| dv_* \right) dv dx \\ &\leq \mathcal{G}^k(g, h) \|f\|_\Lambda,\end{aligned}$$

with

$$\mathcal{G}^k(g, h) = C \sum_{|j_1|+|l_1|+|j_2|+|l_2| \leq k} \left[\int_{\mathbb{T}^N} \|\partial_{l_2}^{j_2} h\|_{\Lambda_v}^2 \|\partial_{l_1}^{j_1} g\|_{L_v^2}^2 dx \right]^{1/2}.$$

At that point we can use Sobolev embeddings (see [4], corollary IX.13) stating that if $E(k_0/2) > N/2$ then we have $H_x^{k/2} \hookrightarrow L_x^\infty$.

So, if $|j_1| + |l_1| \leq k/2$ we have

$$\begin{aligned}(\text{A.1}) \quad \|\partial_{l_1}^{j_1} g\|_{L_v^2}^2 &\leq \sup_{x \in \mathbb{T}^N} \|\partial_{l_1}^{j_1} g\|_{L_v^2}^2 \leq C_k \left\| \|\partial_{l_1}^{j_1} g\|_{L_v^2} \right\|_{H_x^{k/2}}^2 \\ &\leq C_k \sum_{|p| \leq k/2} \sum_{p_1+p_2=p} \int_{\mathbb{T}^N \times \mathbb{R}^N} \partial_{l_1+p_1}^{j_1} g \partial_{l_1+p_2}^{j_1} g dv dx \\ &\leq C_k \|g\|_{H_{x,v}^k}^2,\end{aligned}$$

by a mere Cauchy-Schwarz inequality.

In the other case, $|j_2| + |l_2| \leq k/2$ and by same calculations we show

$$\|\partial_{l_2}^{j_2} h\|_{\Lambda_v}^2 \leq C_k \|h\|_{H_\Lambda^k}^2.$$

Therefore, by just dividing the sum into this two subcases we obtain the result (H4) for Γ^- , noticing that in the case $j = 0$ equation (A.1) has no v derivatives and the Cauchy-Schwarz inequality does not create such derivatives so the control is only made by x -derivatives.

The second term Γ^+ is dealt exactly the same way with, at the end (the study of \mathcal{G}^k), another change of variable $(v, v_*) \rightarrow (v', v'_*)$ which gives the result since $(1 + |v'|)^\gamma \leq (1 + |v|)^\gamma + (1 + |v_*|)^\gamma$ if $\gamma > 0$.

A.3. Landau operator with hard and moderately soft potential. The Landau operator is used to describe plasmas and for instance in the case of particles interacting via a Coulomb interaction (see [28] for more details). The particular case of Coulomb interaction alone ($\gamma = -3$) will not be studied here as the Landau linear operator has a spectral gap if and only if $\gamma \geq 2$ (see [13], for not constructive arguments, [21] for general constructive case and [2] for explicit construction in the case of hard potential $\gamma > 0$) and so only the case $\gamma \geq 2$ may be applicable in this study.

We can compute straightforwardly the bilinear symmetric operator associated with the Landau equation:

$$\Gamma(g, h) = \frac{1}{2\sqrt{M}} \nabla_v \cdot \int_{\mathbb{R}^N} \sqrt{MM_*} \Phi(v - v_*) [g_* \nabla_v h + h_* \nabla_v g - g(\nabla_v h)_* - h(\nabla_v g)_*] dv_*,$$

where $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is such that $\Phi(z)$ is the orthogonal projection onto $\text{Span}(z)^\perp$ so

$$\Phi(z)_{ij} = \delta_{ij} - \frac{z_i z_j}{|z|^2},$$

and γ belongs to $[-2, 1]$.

A.3.1. Orthogonality to $\text{Ker}(L)$: (H5). Let consider a function ψ in $C_{x,v}^\infty$. A mere integration by part gives us

$$\langle \Gamma(g, h), \psi \rangle_{L_v^2} = -\frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \nabla_v \left(\frac{\psi}{\sqrt{M}} \right) \cdot \left(\sqrt{MM_*} \Phi(v - v_*) [G] \right) dv_* dv,$$

where

$$G = g_* \nabla_v h + h_* \nabla_v g - g(\nabla_v h)_* - h(\nabla_v g)_*.$$

Then the change of variable $(v, v_*) \rightarrow (v_*, v)$ only changes $\nabla_v(\psi/\sqrt{M})$ to $\left[\nabla_v(\psi/\sqrt{M}) \right]_*$ and G becomes $-G$. Therefore we finally obtain

$$\langle \Gamma(g, h), \psi \rangle_{L_v^2} = \frac{1}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N} \sqrt{MM_*} \Phi(v - v_*) [G] \cdot \left[\left(\nabla_v \left(\frac{\psi}{\sqrt{M}} \right) \right)_* - \nabla_v \left(\frac{\psi}{\sqrt{M}} \right) \right] dv_* dv.$$

As shown in [20] or [7] we have that $\text{Ker}(L) = \text{Span}(1, v_1, \dots, v_N, |v|^2) M^{1/2}$. Computing the term inside brackets for each of these functions gives us 0 or, in the case $|v|^2 \sqrt{M}$, $2(v_* - v)$.

However, by definition, $\Phi(v - v_*)[G]$ belongs to $\text{Span}(v - v_*)^\perp$ and therefore $\Phi(v - v_*)[G] \cdot (v_* - v) = 0$. So Γ indeed satisfies (H5).

A.3.2. *Controlling derivatives: (H_4).* The article [13] gives us directly the expected result in its Theorem 3, equation (35) with $\theta = 0$. The case where there are only x -derivatives is also included if one takes $\beta = 0$.

APPENDIX B. APPENDIX 2: PROOFS GIVEN THE RESULTS IN THE TOOLBOX

We used the estimates given by the toolbox throughout this article. This annex is to prove all of them. It is divided in two parts. The first one is dedicated to the proof of the equality between null spaces whereas the second part deals with the time derivatives inequalities.

B.1. Proof of proposition 3.1: $\text{Ker}(G) = \text{Ker}(L)$. We are about to prove the following proposition.

Proposition B.1. *Let a and b be in \mathbb{R}^* and consider the operator $G = aL - bv \cdot \nabla_x$ acting on $H_{x,v}^1$. If L satisfies (H1) and (H3) then*

$$\text{Ker}(G) = \text{Ker}(L).$$

To prove this result we will need a lemma.

Lemma B.2. *Let $f : \mathbb{T}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ be continuous on $\mathbb{T}^N \times \mathbb{R}^N$ and differentiable in x .*

If $v \cdot \nabla_x f(x, v) = 0$ for all (x, v) in $\mathbb{T}^N \times \mathbb{R}^N$ then f does not depend on x .

Proof of Lemma B.2. Fix x in \mathbb{T}^N and v \mathbb{Q} -free in \mathbb{R}^N .

For y in \mathbb{R}^N we will denote by \overline{y} its equivalent class in \mathbb{T}^N .

$$\begin{aligned} \text{Define } g : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto f(\overline{x + tv}, v) \end{aligned}$$

We find easily that g is differentiable on \mathbb{R} and that $g'(t) = v \cdot \nabla_x f(x, v) = 0$ on \mathbb{R} . Therefore:

$$\forall t \in \mathbb{R}, f(\overline{x + tv}, v) = f(x, v).$$

However, a well-known property about the torus is that the set $\{x + nv, n \in \mathbb{Z}\}$ is dense in \mathbb{T}^N for all x in \mathbb{T}^N and v \mathbb{Q} -free in \mathbb{R}^N . This combined with the last result and the continuity of f leads to:

$$\forall y \in \mathbb{T}^N, f(y, v) = f(x, v).$$

To conclude it is enough to see that the set of \mathbb{Q} -free vector in \mathbb{R}^N is dense in \mathbb{R}^N and then, by continuity of f in v :

$$\forall y \in \mathbb{T}^N, \forall v \in \mathbb{R}^N, f(y, v) = f(x, v).$$

□

Now we have all the tools to prove the proposition about the Kernel of operators.

Proof of Proposition B.1. Since L satisfies (H1) we know that L acts on L_v^2 and that its Kernel functions ϕ_i only depend on v . Thus, we have directly the first inclusion

$$\text{Ker}(L) \subset \text{Ker}(G).$$

Then, let us consider h in $H_{x,v}^1$ such that $G(h) = 0$. Because the transport operator $v \cdot \nabla_x$ is skew-symmetric in $L_{x,v}^2$ we have

$$0 = \langle G(h), h \rangle_{L_{x,v}^2} = a \int_{\mathbb{T}^N} \langle L(h), h \rangle_{L_v^2} dx.$$

However, because L satisfies (H3) we obtain:

$$0 \geq \lambda \int_{\mathbb{T}^N} \|h(x, \cdot) - \pi_L(h(x, \cdot))\|_{\Lambda_v}^2 dx.$$

But λ is strictly positive and thus:

$$\forall x \in \mathbb{T}^N, \quad h(x, \cdot) = \pi_L(h(x, \cdot)) = \sum_{i=1}^d c_i(x) \phi_i.$$

Finally we have, by assumption, $G(h) = 0$ and because $h(x, \cdot)$ belongs to $\text{Ker}(L)$ for all x in \mathbb{T}^N we end up with

$$\forall (x, v) \in \mathbb{T}^N \times \mathbb{R}^N, \quad v \cdot \nabla_x h(x, v) = 0.$$

By applying the lemma above we then obtain that h does not depend on x . But $(\phi_i)_{1 \leq i \leq d}$ is an orthonormal family, basis of $\text{Ker}(L)$, and therefore we find that for all i , c_i does not depend on x .

So, we have proved that:

$$\forall (x, v) \in \mathbb{T}^N \times \mathbb{R}^N, \quad h(x, v) = \sum_{i=1}^d c_i \phi_i(v).$$

Therefore, h belongs to $\text{Ker}(L)$.

$$\text{Ker}(G) \subset \text{Ker}(L).$$

□

B.2. A priori energy estimates. In this subsection we derive all the inequalities we used. Therefore, we assume that L satisfies (H1'), (H2') and (H3) while Γ has the properties (H4) and (H5), and we pick g in $H_{x,v}^k$. We consider h in $H_{x,v}^k \cap \text{Ker}(G_\varepsilon)^\perp$ and we assume that h is a solution to (1.3):

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon} \Gamma(g, h).$$

In the toolbox, we wrote inequalities on function which were solutions of the linear equation. As the reader may notice, we will deal with the second order operator just by applying the first part of (H4) and Young's inequality. Such an inequality only provides two positive terms, and thus by just setting Γ equal to 0 in the next inequalities we get the expected bounds in the linear case (not the sharpest ones though). Therefore we will just describe the more general case and the linear one is included in it.

B.2.1. Time evolution of pure x -derivatives. The operators L and Γ only act on the v variable. Thus, for $0 \leq |l| \leq k$, ∂_l^0 commutes with L and $v \cdot \nabla_x$. Remind that $v \cdot \nabla_x$ is skew-symmetric in $L_{x,v}^2(\mathbb{T}^N \times \mathbb{R}^N)$ and therefore we can compute

$$\frac{d}{dt} \|\partial_l^0 h\|_{L_{x,v}^2}^2 = \frac{2}{\varepsilon^2} \langle L(\partial_l^0 h), \partial_l^0 h \rangle_{L_{x,v}^2} + \frac{1}{\varepsilon} \langle \partial_l^0 \Gamma(g, h), \partial_l^0 h \rangle_{L_{x,v}^2}.$$

We can then use hypothesis (H3) to obtain

$$\frac{2}{\varepsilon^2} \langle L(\partial_l^0 h), \partial_l^0 h \rangle_{L_{x,v}^2} \leq -\frac{2\lambda}{\varepsilon^2} \|(\partial_l^0 h)^\perp\|_\Lambda^2.$$

We also use (H3) to get $(\partial_l^0 h)^\perp = \partial_l^0 h^\perp$.

To deal with the second scalar product, we will use hypothesis (H4) and (H5), which is still valid for $\partial_l^0 \Gamma$ since π_L only acts on the v variable, followed by a Young inequality with some $D_1 > 0$. This yields

$$\begin{aligned} \frac{2}{\varepsilon} \langle \partial_l^0 \Gamma(g, h), \partial_l^0 h \rangle_{L_{x,v}^2} &= \frac{2}{\varepsilon} \langle \partial_l^0 \Gamma(g, h), \partial_l^0 h^\perp \rangle_{L_{x,v}^2} \\ &\leq \frac{2}{\varepsilon} \mathcal{G}_x^k(g, h) \|\partial_l^0 h^\perp\|_\Lambda \\ &\leq \frac{D_1}{\varepsilon} (\mathcal{G}_x^k(g, h))^2 + \frac{1}{D_1 \varepsilon} \|\partial_l^0 h^\perp\|_\Lambda^2. \end{aligned}$$

Gathering the last two upper bounds we give raise to

$$\frac{d}{dt} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \leq \left[\frac{1}{D_1 \varepsilon} - \frac{2\lambda}{\varepsilon^2} \right] \|\partial_l^0 h^\perp\|_\Lambda^2 + \frac{D_1}{\varepsilon} (\mathcal{G}_x^k(g, h))^2.$$

Finally, taking $D_1 = \varepsilon/\lambda$ gives us inequalities (3.6), (3.7) and (3.10).

B.2.2. Time evolution of $\|\nabla_v h\|_{L_{x,v}^2}^2$. For that term we get, by applying the equation satisfied by h , the following:

$$\frac{d}{dt} \|\nabla_v h\|_{L_{x,v}^2}^2 = \frac{2}{\varepsilon^2} \langle \nabla_v L(h), \nabla_v h \rangle_{L_{x,v}^2} - \frac{2}{\varepsilon} \langle \nabla_v (v \cdot \nabla_x h), \nabla_v h \rangle_{L_{x,v}^2} + \frac{2}{\varepsilon} \langle \nabla_v \Gamma(g, h), \nabla_v h \rangle_{L_{x,v}^2}.$$

And by writing the second term on the right-hand side of the equality and integrating by part in x , we raise

$$\langle \nabla_v (v \cdot \nabla_x h), \nabla_v h \rangle_{L_{x,v}^2} = \langle \nabla_x h, \nabla_v h \rangle_{L_{x,v}^2}.$$

Therefore the following holds:

$$\frac{d}{dt} \|\nabla_v h\|_{L_{x,v}^2}^2 = \frac{2}{\varepsilon^2} \langle \nabla_v L(h), \nabla_v h \rangle_{L_{x,v}^2} - \frac{2}{\varepsilon} \langle \nabla_x h, \nabla_v h \rangle_{L_{x,v}^2} + \frac{2}{\varepsilon} \langle \nabla_v \Gamma(g, h), \nabla_v h \rangle_{L_{x,v}^2}.$$

Then we have by (H1) that $L = K - \Lambda$ and we can estimate each component thanks to (H1) and (H2):

$$\begin{aligned} -\langle \nabla_v \Lambda(h), \nabla_v h \rangle_{L_{x,v}^2} &\leq \nu_4^\Lambda \|h\|_{L_{x,v}^2}^2 - \nu_3^\Lambda \|\nabla_v h\|_\Lambda^2, \\ \langle \nabla_v K(h), \nabla_v h \rangle_{L_{x,v}^2} &\leq C(\delta) \|h\|_{L_{x,v}^2}^2 + \delta \|\nabla_v h\|_{L_{x,v}^2}^2, \end{aligned}$$

where δ is a strictly positive real that we will choose later.

Finally, for a $D > 0$ that we will choose later, we have the following upper bound, by Cauchy-Schwarz inequality:

$$-\frac{2}{\varepsilon} \langle \nabla_x h, \nabla_v h \rangle_{L^2_{x,v}} \leq \frac{D}{\varepsilon} \|\nabla_x h\|_{L^2_{x,v}}^2 + \frac{\nu_1^\Lambda}{D\nu_0^\Lambda \varepsilon} \|\nabla_v h\|_\Lambda^2,$$

using the fact that $\|\cdot\|_{L^2_{x,v}}^2 \leq \frac{\nu_1^\Lambda}{\nu_0^\Lambda} \|\cdot\|_\Lambda^2$. Finally, another Young inequality gives us a control on the last scalar product, for a $D_2 > 0$ to be chosen later

$$\frac{2}{\varepsilon} \langle \nabla_v \Gamma(g, h), \nabla_v h \rangle_{L^2_{x,v}} \leq \frac{D_2}{\varepsilon} (\mathcal{G}_{x,v}^1(g, h))^2 + \frac{1}{D_2 \varepsilon} \|\nabla_v h\|_\Lambda^2.$$

We gather here the last three inequalities to obtain our global upper bound:

$$\begin{aligned} \frac{d}{dt} \|\nabla_v h\|_{L^2_{x,v}}^2 &\leq \frac{1}{\varepsilon^2} (2\nu_4^\Lambda + 2C(\delta)) \|h\|_{L^2_{x,v}}^2 + \frac{D}{\varepsilon} \|\nabla_x h\|_{L^2_{x,v}}^2 \\ &\quad + \left(\frac{2\nu_1^\Lambda \delta}{\nu_0^\Lambda \varepsilon^2} - \frac{2\nu_3^\Lambda}{\varepsilon^2} + \frac{\nu_1^\Lambda}{D\varepsilon\nu_0^\Lambda} + \frac{1}{D_2 \varepsilon} \right) \|\nabla_v h\|_\Lambda^2 + \frac{D_2}{\varepsilon} (\mathcal{G}_{x,v}^1(g, h))^2. \end{aligned}$$

We can go even further since we have $\|h\|_{L^2_{x,v}}^2 = \|h^\perp\|_{L^2_{x,v}}^2 + \|\pi_L(h)\|_{L^2_{x,v}}^2$.

But because h is in $\text{Ker}(G_\varepsilon)^\perp$ we can use the toolbox and the equation (3.5) about the Poincaré inequality:

$$\|\pi_L(h)\|_{L^2_{x,v}}^2 \leq C_p \|\nabla_x h\|_{L^2_{x,v}}^2.$$

This last inequality yields:

$$\begin{aligned} \frac{d}{dt} \|\nabla_v h\|_{L^2_{x,v}}^2 &\leq \frac{\nu_1^\Lambda}{\nu_0^\Lambda \varepsilon^2} (2\nu_4^\Lambda + 2C(\delta)) \|h^\perp\|_\Lambda^2 + \left[\frac{C_p}{\varepsilon^2} (2\nu_4^\Lambda + 2C(\delta)) + \frac{D}{\varepsilon} \right] \|\nabla_x h\|_{L^2_{x,v}}^2 \\ &\quad + \left[\frac{2\nu_1^\Lambda \delta}{\nu_0^\Lambda \varepsilon^2} - \frac{2\nu_3^\Lambda}{\varepsilon^2} + \frac{\nu_1^\Lambda}{D\varepsilon\nu_0^\Lambda} + \frac{1}{D_2 \varepsilon} \right] \|\nabla_v h\|_\Lambda^2 + \frac{D_2}{\varepsilon} (\mathcal{G}_{x,v}^1(g, h))^2. \end{aligned}$$

Therefore, we can choose $\delta = \nu_0^\Lambda \nu_3^\Lambda / 6\nu_1^\Lambda$, $D = 3\nu_1^\Lambda \varepsilon / \nu_0^\Lambda \nu_3^\Lambda$ and $D_2 = 3\varepsilon / \nu_3^\Lambda$ to get the equation (3.8).

B.2.3. Time evolution of $\langle \nabla_x h, \nabla_v h \rangle_{L^2_{x,v}}$. In the same way, and integrating by part in x then in v we obtain the following equality:

$$\frac{d}{dt} \langle \nabla_x h, \nabla_v h \rangle_{L^2_{x,v}} = \frac{2}{\varepsilon^2} \langle L(\nabla_x h), \nabla_v h \rangle_{L^2_{x,v}} - \frac{2}{\varepsilon} \langle \nabla_v(v \cdot \nabla_x h), \nabla_x h \rangle_{L^2_{x,v}} + \frac{2}{\varepsilon} \langle \nabla_x \Gamma(g, h), \nabla_v h \rangle_{L^2_{x,v}}.$$

By writing explicitly $\langle \nabla_v(v \cdot \nabla_x h), \nabla_x h \rangle_{L^2_{x,v}}$ and by integrating by part one can show that the following holds:

$$\langle \nabla_v(v \cdot \nabla_x h), \nabla_x h \rangle_{L^2_{x,v}} = \frac{1}{2} \|\nabla_x h\|_{L^2_{x,v}}^2.$$

Therefore we have an explicit formula for that term and we can find the time derivative of the scalar product being:

$$\frac{d}{dt} \langle \nabla_x h, \nabla_v h \rangle_{L^2_{x,v}} = \frac{2}{\varepsilon^2} \langle L(\nabla_x h), \nabla_v h \rangle_{L^2_{x,v}} - \frac{1}{\varepsilon} \|\nabla_x h\|_{L^2_{x,v}}^2 + \frac{2}{\varepsilon} \langle \nabla_x \Gamma(g, h), \nabla_v h \rangle_{L^2_{x,v}}.$$

We can bound above the first term in the right-hand side of the equality thanks to (H1) and then Cauchy-Schwarz in x , with a constant $\eta > 0$ to be define later.

$$\begin{aligned}
\frac{2}{\varepsilon^2} \langle L(\nabla_x h), \nabla_v h \rangle_{L_{x,v}^2} &= \frac{2}{\varepsilon^2} \langle L(\nabla_x h^\perp), \nabla_v h \rangle_{L_{x,v}^2} \\
&\leq \frac{C^L}{\varepsilon^2} \int_{\mathbb{T}^N} 2 \|\nabla_x h^\perp\|_{\Lambda_v} \|\nabla_v h\|_{\Lambda_v} dx \\
&\leq \frac{C^L \eta}{\varepsilon^2} \|\nabla_x h^\perp\|_{\Lambda}^2 + \frac{C^L}{\eta \varepsilon^2} \|\nabla_v h\|_{\Lambda}^2.
\end{aligned}$$

Then applying hypothesis (H4) and Young's inequality one more time with a constant $D_3 > 0$ one may find

$$\frac{2}{\varepsilon} \langle \nabla_x \Gamma(g, h), \nabla_v h \rangle_{L_{x,v}^2} \leq \frac{D_3}{\varepsilon} (\mathcal{G}_x^1(g, h))^2 + \frac{1}{D_3 \varepsilon} \|\nabla_v h\|_{\Lambda}^2.$$

Hence we end up with the following inequality:

$$\begin{aligned}
\frac{d}{dt} \langle \nabla_x h, \nabla_v h \rangle_{L_{x,v}^2} &\leq \frac{C^L \eta}{\varepsilon^2} \|\nabla_x h^\perp\|_{\Lambda}^2 + \left(\frac{C^L}{\eta \varepsilon^2} + \frac{1}{D_3} \right) \|\nabla_v h\|_{\Lambda}^2 - \frac{1}{\varepsilon} \|\nabla_x h\|_{L_{x,v}^2}^2 \\
&\quad + \frac{D_3}{\varepsilon} (\mathcal{G}_x^1(g, h))^2.
\end{aligned}$$

Now define $\eta = e/\varepsilon$, $e > 0$, and $D_3 = e/C^L$ to obtain equation (3.9).

B.2.4. Time evolution of $\|\partial_l^j h\|_{L_{x,v}^2}^2$ for $|j| \geq 1$ and $|j| + |l| = k$. This term is the only term far from what we already did since we are mixing more than one derivative in x and one derivative in v in general. By simply differentiating in time and integrating by part we find the following equality.

$$\begin{aligned}
\frac{d}{dt} \|\partial_l^j h\|_{L_{x,v}^2}^2 &= \frac{2}{\varepsilon^2} \langle \partial_l^j L(h), \partial_l^j h \rangle_{L_{x,v}^2} - \frac{2}{\varepsilon} \langle \partial_l^j (v \cdot \nabla_x h), \partial_l^j h \rangle_{L_{x,v}^2} \\
&\quad + \frac{2}{\varepsilon} \langle \partial_l^j \Gamma(g, h), \partial_l^j h \rangle_{L_{x,v}^2} \\
&= \frac{2}{\varepsilon^2} \langle \partial_l^j L(h), \partial_l^j h \rangle_{L_{x,v}^2} - \frac{2}{\varepsilon} \sum_{i, c_i(j) > 0} \langle \partial_l^j h, \partial_{l+\delta_i}^{j-\delta_i} h \rangle_{L_{x,v}^2} \\
&\quad + \frac{2}{\varepsilon} \langle \partial_l^j \Gamma(g, h), \partial_l^j h \rangle_{L_{x,v}^2}.
\end{aligned}$$

We can then apply Cauchy-Schwarz for the terms inside the sum symbol. For each we can use a $D_{i,l,k} > 0$ but because they play an equivalent role we will take the same $D > 0$, that we will choose later:

$$-\frac{2}{\varepsilon} \langle \partial_l^j h, \partial_{l+\delta_i}^{j-\delta_i} h \rangle_{L_{x,v}^2} \leq \frac{\nu_1^\Lambda}{D \nu_0^\Lambda \varepsilon} \|\partial_l^j h\|_{\Lambda}^2 + \frac{D}{\varepsilon} \|\partial_{l+\delta_i}^{j-\delta_i} h\|_{L_{x,v}^2}^2.$$

Then we can use (H1') and (H2'), with a $\delta > 0$ we will choose later, to obtain

$$\frac{2}{\varepsilon^2} \langle \partial_l^j L(h), \partial_l^j h \rangle_{L_{x,v}^2} \leq \frac{2}{\varepsilon^2} (C(\delta) + \nu_6^\Lambda) \|h\|_{H_{x,v}^{k-1}}^2 + \frac{2}{\varepsilon^2} \left(\frac{\delta \nu_1^\Lambda}{\nu_0^\Lambda} - \nu_5^\Lambda \right) \|\partial_l^j h\|_{\Lambda}^2.$$

Finally, applying (H4) and Young's inequality with a constant $D_2 > 0$ we raise

$$\frac{2}{\varepsilon} \langle \partial_l^j \Gamma(g, h), \partial_l^j h \rangle_{L_{x,v}^2} \leq \frac{D_2}{\varepsilon} (\mathcal{G}_{x,v}^k(g, h))^2 + \frac{1}{D_2 \varepsilon} \|\partial_l^j h\|_{\Lambda}^2.$$

Combining these three inequality we find an upper bound for the time evolution. Here we also use the fact that the number of i such that $c_i(j) > 0$ is less or equal to N .

$$\begin{aligned} \frac{d}{dt} \|\partial_l^j h\|_{L_{x,v}^2}^2 &\leq \left[\frac{\nu_1^\Lambda N}{D\varepsilon\nu_0^\Lambda} + \frac{2}{\varepsilon^2} \left(\frac{\delta\nu_1^\Lambda}{\nu_0^\Lambda} - \nu_5^\Lambda \right) + \frac{1}{D_2\varepsilon} \right] \|\partial_l^j h\|_\Lambda^2 \\ &\quad + \frac{D}{\varepsilon} \sum_{i, c_i(j) > 0} \left\| \partial_{l+\delta_i}^{j-\delta_i} h \right\|_{L_{x,v}^2}^2 + \frac{2}{\varepsilon^2} (C(\delta) + \nu_6^\Lambda) \|h\|_{H_{x,v}^{k-1}}^2 \\ &\quad + \frac{D_2}{\varepsilon} (\mathcal{G}_{x,v}^k(g, h))^2. \end{aligned}$$

Hence, we obtain equations (3.11) and (3.12) by taking $D = 3\nu_1^\Lambda\varepsilon/\nu_0^\Lambda\nu_5^\Lambda$, $D_2 = 3\varepsilon/\nu_5^\Lambda$ and $\delta = \nu_0^\Lambda\nu_5^\Lambda/6\nu_1^\Lambda$. Also note that in (3.11) we used $\left\| \partial_{l+\delta_i}^{j-\delta_i} h \right\|_{L_{x,v}^2}^2 \leq \frac{\nu_1^\Lambda}{\nu_0^\Lambda} \left\| \partial_{l+\delta_i}^{j-\delta_i} h \right\|_\Lambda^2$.

B.2.5. Time evolution of $\langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2}$. With no more calculations, we can bound this term in the same way we did for $\frac{d}{dt} \langle \nabla_x h, \nabla_v h \rangle_{L_{x,v}^2}$. Here we get

$$\begin{aligned} \frac{d}{dt} \langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2} &\leq \frac{C^L \eta}{\varepsilon^2} \|\partial_l^0 h^\perp\|_\Lambda^2 + \left[\frac{C^L}{\eta\varepsilon^2} + \frac{1}{\varepsilon D_3} \right] \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2 - \frac{1}{\varepsilon} \|\partial_l^0 h\|_{L_{x,v}^2}^2 \\ &\quad + \frac{D_3}{\varepsilon} (\mathcal{G}_x^k(g, h))^2. \end{aligned}$$

Now define $\eta = e/\varepsilon$, $e > 0$, and $D_3 = e/C^L$ to obtain equation (3.13).

In the next paragraphs, we are setting $g = h$.

B.2.6. Time evolution of $\|\nabla_v h^\perp\|_{L_{x,v}^2}^2$. By simply differentiating norm and using (H5) to get $\Gamma(h, h)^\perp = \Gamma(h, h)$, we compute

$$\frac{d}{dt} \|\nabla_v h^\perp\|_{L_{x,v}^2}^2 = 2 \langle \nabla_v (G_\varepsilon(h))^\perp, \nabla_v h^\perp \rangle_{L_{x,v}^2} + \frac{2}{\varepsilon} \langle \nabla_v \Gamma(h, h), \nabla_v h^\perp \rangle_{L_{x,v}^2}.$$

By applying (H4) and Young's inequality to the second term on the right-hand side, with a constant $D_2 > 0$, and controlling the $L_{x,v}^2$ -norm by the Λ -norm we obtain:

$$\frac{2}{\varepsilon} \langle \nabla_v \Gamma(h, h), \nabla_v h^\perp \rangle_{L_{x,v}^2} \leq \frac{D_2}{\varepsilon} (\mathcal{G}_{x,v}^1(h, h))^2 + \frac{1}{\varepsilon D_2} \|\nabla_v h^\perp\|_\Lambda^2.$$

Then we have to control the first term. Just by writing it and decomposing terms in projection onto $\text{Ker}(L)$ and onto its orthogonal we yield:

$$\begin{aligned} 2 \langle \nabla_v (G_\varepsilon(h))^\perp, \nabla_v h^\perp \rangle_{L_{x,v}^2} &= \frac{2}{\varepsilon^2} \langle \nabla_v L(h), \nabla_v h^\perp \rangle_{L_{x,v}^2} - \frac{2}{\varepsilon} \langle \nabla_v (v \cdot \nabla_x h)^\perp, \nabla_v h^\perp \rangle_{L_{x,v}^2} \\ &= \frac{2}{\varepsilon^2} \langle \nabla_v L(h^\perp), \nabla_v h^\perp \rangle_{L_{x,v}^2} - \frac{2}{\varepsilon} \langle \nabla_x h, \nabla_v h^\perp \rangle_{L_{x,v}^2} \\ &\quad - \frac{2}{\varepsilon} \langle v \cdot \nabla_v \nabla_x \pi_L(h), \nabla_v h^\perp \rangle_{L_{x,v}^2} \\ &\quad + \frac{2}{\varepsilon} \langle \nabla_v \pi_L(v \cdot \nabla_x h), \nabla_v h^\perp \rangle_{L_{x,v}^2}. \end{aligned}$$

Then we can control the first term on the right-hand side thanks to (H1) and (H2), $\delta > 0$ to be chosen later:

$$\frac{2}{\varepsilon^2} \langle \nabla_v L(h^\perp), \nabla_v h^\perp \rangle_{L_{x,v}^2} \leq \frac{2(C(\delta) + \nu_4^\Lambda) \nu_1^\Lambda}{\nu_0^\Lambda \varepsilon^2} \|h^\perp\|_\Lambda^2 + \frac{2}{\varepsilon^2} \left(\frac{\nu_1^\Lambda \delta}{\nu_0^\Lambda} - \nu_3^\Lambda \right) \|\nabla_v h^\perp\|_\Lambda^2.$$

We apply Cauchy-Schwarz inequality to the next term, with D to be chosen later:

$$-\frac{2}{\varepsilon} \langle \nabla_x h, \nabla_v h^\perp \rangle_{L_{x,v}^2} \leq \frac{D}{\varepsilon} \|\nabla_x h\|_{L_{x,v}^2}^2 + \frac{\nu_1^\Lambda}{\nu_0^\Lambda D \varepsilon} \|\nabla_v h^\perp\|_\Lambda^2.$$

For the third term we are going to apply Cauchy-Schwarz inequality and then use the property (H3). The latter property tells us that the functions in $\text{Ker}(L)$ are of the form a polynomial in v times $e^{-|v|^2/4}$. This fact combined with the shape of π_L , equation (3.1), shows us that we can control, by a mere Cauchy-Schwarz inequality, the third term. Then the property (3.3) yields the following upper bound:

$$\begin{aligned} -\frac{2}{\varepsilon} \langle v \cdot \nabla_v \nabla_x \pi_L(h), \nabla_v h^\perp \rangle_{L_{x,v}^2} &\leq \frac{\tilde{D}}{\varepsilon} \|v \cdot \nabla_v \pi_L(\nabla_x h)\|_{L_{x,v}^2}^2 + \frac{1}{\tilde{D} \varepsilon} \|\nabla_v h^\perp\|_{L_{x,v}^2}^2 \\ &\leq \frac{\tilde{D} C_{\pi 1}}{\varepsilon} \|\nabla_x h\|_{L_{x,v}^2}^2 + \frac{\nu_1^\Lambda}{\nu_0^\Lambda \tilde{D} \varepsilon} \|\nabla_v h^\perp\|_\Lambda^2. \end{aligned}$$

Finally, we first use equation (3.3) controlling the v -derivatives of π_L and then see that the norm of $\pi_L(v \cdot f)$ is easily controlled by the norm of f (just use (H3) and the definition of π_L (3.1) and apply Cauchy-Schwarz inequality) by a factor $C_{\pi 1}$ (increase this constant if necessary in (3.3)):

$$\begin{aligned} \frac{2}{\varepsilon} \langle \nabla_v \pi_L(v \cdot \nabla_x h), \nabla_v h^\perp \rangle_{L_{x,v}^2} &\leq \frac{D'}{\varepsilon} \|\nabla_v \pi_L(v \cdot \nabla_x h)\|_{L_{x,v}^2}^2 + \frac{1}{\varepsilon D'} \|\nabla_v h^\perp\|_{L_{x,v}^2}^2 \\ &\leq \frac{D' C_{\pi 1}}{\varepsilon} \|\pi_L(v \cdot \nabla_x h)\|_{L_{x,v}^2}^2 + \frac{1}{\varepsilon D'} \|\nabla_v h^\perp\|_{L_{x,v}^2}^2 \\ &\leq \frac{D' C_{\pi 1}^2}{\varepsilon} \|\nabla_x h\|_{L_{x,v}^2}^2 + \frac{\nu_1^\Lambda}{\nu_0^\Lambda \varepsilon D'} \|\nabla_v h^\perp\|_{L_{x,v}^2}^2. \end{aligned}$$

We then gather all those bounds to get the last upper bound for the time derivative of the v -derivative.

$$\begin{aligned} \frac{d}{dt} \|\nabla_v h^\perp\|_{L_{x,v}^2}^2 &\leq \frac{\nu_1^\Lambda}{\nu_0^\Lambda \varepsilon^2} (2\nu_4^\Lambda + 2C(\delta)) \|h^\perp\|_\Lambda^2 + \left[\frac{D}{\varepsilon} + \frac{D' C_{\pi 1}^2}{\varepsilon} + \frac{\tilde{D} C_{\pi 1}}{\varepsilon} \right] \|\nabla_x h\|_{L_{x,v}^2}^2 \\ &\quad + \left[\frac{2\nu_1^\Lambda \delta}{\nu_0^\Lambda \varepsilon^2} - \frac{2\nu_3^\Lambda}{\varepsilon^2} + \frac{\nu_1^\Lambda}{\varepsilon \nu_0^\Lambda} \left(\frac{1}{D} + \frac{1}{D'} + \frac{1}{\tilde{D}} \right) + \frac{1}{\varepsilon D_2} \right] \|\nabla_v h^\perp\|_\Lambda^2 \\ &\quad + \frac{D_2}{\varepsilon} (\mathcal{G}_{x,v}^1(h, h))^2. \end{aligned}$$

Therefore we obtain (3.14) by taking $D = D' = \tilde{D} = 9\nu_1^\Lambda \varepsilon / \nu_0^\Lambda \nu_3^\Lambda$, $\delta = \nu_0^\Lambda \nu_3^\Lambda / 6\nu_1^\Lambda$ and $D_2 = 3\varepsilon / \nu_3^\Lambda$.

B.2.7. *A new time evolution of $\langle \nabla_x h, \nabla_v h \rangle_{L^2_{x,v}}$.* By integrating by part in x then in v we obtain the following equality on the evolution of the scalar product.

$$\frac{d}{dt} \langle \nabla_x h, \nabla_v h \rangle_{L^2_{x,v}} = 2 \langle \nabla_x G_\varepsilon(h), \nabla_v h \rangle_{L^2_{x,v}} + \frac{2}{\varepsilon} \langle \nabla_v \Gamma(h, h), \nabla_x h \rangle_{L^2_{x,v}}.$$

We will bound above the first term as in the previous case and for the second term involving Γ we use (H4) and Young's inequality with a constant $D_3 > 0$:

$$2 \langle \nabla_v \Gamma(h, h), \nabla_x h \rangle_{L^2_{x,v}} \leq D_3 (\mathcal{G}_{x,v}^1(h, h))^2 + \frac{1}{D_3} \|\nabla_x h\|_\Lambda^2.$$

We decompose $\nabla_x h$ thanks to π_L and we use (3.4) to control the fluid part of it,

$$2 \langle \nabla_v \Gamma(h, h), \nabla_x h \rangle_{L^2_{x,v}} \leq D_3 (\mathcal{G}_{x,v}^1(h, h))^2 + \frac{1}{D_3} \|\nabla_x h^\perp\|_\Lambda^2 + \frac{C_\pi}{D_3} \|\nabla_x h\|_{L^2_{x,v}}^2.$$

Finally we obtain an upper bound for the time-derivative:

$$\begin{aligned} \frac{d}{dt} \langle \nabla_x h, \nabla_v h \rangle_{L^2_{x,v}} &\leq \left[\frac{C^L \eta}{\varepsilon^2} + \frac{1}{\varepsilon D_3} \right] \|\nabla_x h^\perp\|_\Lambda^2 + \frac{C^L}{\eta \varepsilon^2} \|\nabla_v h\|_\Lambda^2 + \left[\frac{C_\pi}{\varepsilon D_3} - \frac{1}{\varepsilon} \right] \|\nabla_x h\|_{L^2_{x,v}}^2 \\ &\quad + \frac{D_3}{\varepsilon} (\mathcal{G}_{x,v}^1(h, h))^2. \end{aligned}$$

But now, we can use the properties (3.3) and (3.4) of the projection π_L to go further.

$$\begin{aligned} \|\nabla_v h\|_\Lambda^2 &\leq 2 \|\nabla_v h^\perp\|_\Lambda^2 + 2 \|\nabla_v \pi_L(h)\|_\Lambda^2 \\ &\leq 2 \|\nabla_v h^\perp\|_\Lambda^2 + 2 C_{\pi 1} C_\pi \|\pi_L(h)\|_{L^2_{x,v}}^2 \\ &\leq 2 \|\nabla_v h^\perp\|_\Lambda^2 + 2 C_{\pi 1} C_\pi C_p \|\nabla_x h\|_{L^2_{x,v}}^2, \end{aligned}$$

where we used Poincare inequality (3.5) because h is in $\text{Ker}(G_\varepsilon)^\perp$.

Hence we have a final upper bound for the time derivative:

$$\begin{aligned} \frac{d}{dt} \langle \nabla_x h, \nabla_v h \rangle_{L^2_{x,v}} &\leq \left[\frac{C^L \eta}{\varepsilon^2} + \frac{1}{\varepsilon D_3} \right] \|\nabla_x h^\perp\|_\Lambda^2 \\ &\quad + \frac{2C^L}{\eta \varepsilon^2} \|\nabla_v h^\perp\|_\Lambda^2 + \left[\frac{2C^L C_{\pi 1} C_\pi C_p}{\varepsilon^2 \eta} + \frac{C_\pi}{\varepsilon D_3} - \frac{1}{\varepsilon} \right] \|\nabla_x h\|_{L^2_{x,v}}^2 \\ &\quad + \frac{D_3}{\varepsilon} (\mathcal{G}_{x,v}^1(h, h))^2. \end{aligned}$$

Thus, setting $\eta = 8eC^L C_{\pi 1} C_\pi C_p / \varepsilon$ with $e \geq 1$ and $D_3 = 4C_\pi$ we obtain equation (3.15).

B.2.8. *Time evolution of $\|\partial_l^j h^\perp\|_{L^2_{x,v}}^2$, $j \geq 1$ and $|j| + |l| = k$.* We have the following time evolution:

$$\frac{d}{dt} \|\partial_l^j h^\perp\|_{L^2_{x,v}}^2 = 2 \langle \partial_l^j (G_\varepsilon(h))^\perp, \partial_l^j h^\perp \rangle_{L^2_{x,v}} + \frac{2}{\varepsilon} \langle \partial_l^j \Gamma(h, h), \partial_l^j h^\perp \rangle_{L^2_{x,v}}.$$

As above, we apply (H4) for the last term on the right hand side, with a constant $D_2 > 0$.

$$2\langle \partial_l^j \Gamma(h, h), \partial_l^j h^\perp \rangle_{L_{x,v}^2} \leq D_2 (\mathcal{G}_{x,v}^k(h, h))^2 + \frac{1}{D_2} \|\partial_l^j h^\perp\|_\Lambda^2.$$

Then we evaluate the first term on the right-hand side.

$$\begin{aligned} 2\langle \partial_l^j (G_\varepsilon(h))^\perp, \partial_l^j h^\perp \rangle_{L_{x,v}^2} &= \frac{2}{\varepsilon^2} \langle \partial_l^j L(h), \partial_l^j h^\perp \rangle_{L_{x,v}^2} - \frac{2}{\varepsilon} \langle \partial_l^j (v \cdot \nabla_x h)^\perp, \partial_l^j h^\perp \rangle_{L_{x,v}^2} \\ &= \frac{2}{\varepsilon^2} \langle \partial_l^j L(h^\perp), \partial_l^j h^\perp \rangle_{L_{x,v}^2} - \frac{2}{\varepsilon} \langle v \cdot \partial_l^j \pi_L(\nabla_x h), \partial_l^j h^\perp \rangle_{L_{x,v}^2} \\ &\quad - \frac{2}{\varepsilon} \sum_{i, c_i(j) > 0} \langle \partial_{l+\delta_i}^{j-\delta_i} h, \partial_l^j h^\perp \rangle_{L_{x,v}^2} \\ &\quad + \frac{2}{\varepsilon} \langle \partial_l^j \pi_L(v \cdot \nabla_x h), \partial_l^j h^\perp \rangle_{L_{x,v}^2}. \end{aligned}$$

Then we shall bound each of these four terms on the right-hand side.

We can first use the properties (H1') and (H2') of L to get, for some δ to be chosen later,

$$\frac{2}{\varepsilon^2} \langle \partial_l^j L(h^\perp), \partial_l^j h^\perp \rangle_{L_{x,v}^2} \leq \frac{2}{\varepsilon^2} (C(\delta) + \nu_6^\Lambda) \|h^\perp\|_{H_{x,v}^{k-1}}^2 + \frac{2}{\varepsilon^2} \left(\frac{\nu_1^\Lambda \delta}{\nu_0^\Lambda} - \nu_5^\Lambda \right) \|\partial_l^j h^\perp\|_\Lambda^2.$$

For the three remaining terms we will apply Cauchy-Schwarz inequality and use the properties of π_L concerning v -derivatives and multiplications by a polynomial in v .

First

$$\begin{aligned} -\frac{2}{\varepsilon} \langle v \cdot \partial_l^j \pi_L(\nabla_x h), \partial_l^j h^\perp \rangle_{L_{x,v}^2} &\leq \frac{D}{\varepsilon} \|v \cdot \partial_l^j \pi_L(\nabla_x h)\|_{L_{x,v}^2}^2 + \frac{1}{D\varepsilon} \|\partial_l^j h^\perp\|_{L_{x,v}^2}^2 \\ &\leq \frac{DC_{\pi k}}{\varepsilon} \|\partial_l^0(\nabla_x h)\|_{L_{x,v}^2}^2 + \frac{\nu_1^\Lambda}{\nu_0^\Lambda D\varepsilon} \|\partial_l^j h^\perp\|_\Lambda^2 \\ &\leq \begin{cases} \frac{DC_{\pi k}}{\varepsilon} \sum_{|l'|=k} \|\partial_{l'}^0 h\|_{L_{x,v}^2}^2 + \frac{\nu_1^\Lambda}{\nu_0^\Lambda D\varepsilon} \|\partial_l^j h^\perp\|_\Lambda^2, & \text{if } |j| = 1 \\ \frac{DC_{\pi k}}{\varepsilon} \sum_{|l'| \leq k-1} \|\partial_{l'}^0 h\|_{L_{x,v}^2}^2 + \frac{\nu_1^\Lambda}{\nu_0^\Lambda D\varepsilon} \|\partial_l^j h^\perp\|_\Lambda^2, & \text{if } |j| > 1, \end{cases} \end{aligned}$$

where we used that $|l| = |k| - |j|$. Then

$$-\frac{2}{\varepsilon} \langle \partial_{l+\delta_i}^{j-\delta_i} h, \partial_l^j h^\perp \rangle_{L_{x,v}^2} \leq \frac{D'}{\varepsilon} \|\partial_{l+\delta_i}^{j-\delta_i} h\|_{L_{x,v}^2}^2 + \frac{\nu_1^\Lambda}{\nu_0^\Lambda D'\varepsilon} \|\partial_l^j h^\perp\|_\Lambda^2$$

In the case where $|j| > 1$ we can also use that $\|\partial_{l+\delta_i}^{j-\delta_i} h\|_{L_{x,v}^2}^2$ can be decomposed thanks to π_L and its orthogonal projector. Then the fluid part is controlled by the x -derivatives only.

And finally

$$\begin{aligned}
\frac{2}{\varepsilon} \langle \partial_l^j \pi_L(v \cdot \nabla_x h), \partial_l^j h^\perp \rangle_{L_{x,v}^2} &\leq \frac{\tilde{D}}{\varepsilon} \|\partial_l^j \pi_L(v \cdot \nabla_x h)\|_{L_{x,v}^2}^2 + \frac{1}{\tilde{D}\varepsilon} \|\partial_l^j h^\perp\|_{L_{x,v}^2}^2 \\
&\leq \frac{\tilde{D}C_{\pi k}}{\varepsilon} \|\partial_l^0 \nabla_x h\|_{L_{x,v}^2}^2 + \frac{\nu_1^\Lambda}{\tilde{D}\nu_0^\Lambda \varepsilon} \|\partial_l^j h^\perp\|_\Lambda^2 \\
&\leq \begin{cases} \frac{\tilde{D}C_{\pi k}}{\varepsilon} \sum_{|l'|=k} \|\partial_{l'}^0 h\|_{L_{x,v}^2}^2 + \frac{\nu_1^\Lambda}{\nu_0^\Lambda \tilde{D}\varepsilon} \|\partial_l^j h^\perp\|_\Lambda^2, & \text{if } |j| = 1 \\ \frac{\tilde{D}C_{\pi k}}{\varepsilon} \sum_{|l'| \leq k-1} \|\partial_{l'}^0 h\|_{L_{x,v}^2}^2 + \frac{\nu_1^\Lambda}{\nu_0^\Lambda \tilde{D}\varepsilon} \|\partial_l^j h^\perp\|_\Lambda^2, & \text{if } |j| > 1, \end{cases}
\end{aligned}$$

We are now able to combine all those estimates to get an upper bound of the time-derivative we are looking at. We can also give to different bounds, depending on the size $|j|$. We also used that the number of i such that $c_i(j) > 0$ is less than N .

In the case $|j| > 1$,

$$\begin{aligned}
\frac{d}{dt} \|\partial_l^j h^\perp\|_{L_{x,v}^2}^2 &\leq \left[\frac{2}{\varepsilon^2} \left(\frac{\nu_1^\Lambda \delta}{\nu_0^\Lambda} - \nu_5^\Lambda \right) + \frac{\nu_1^\Lambda}{\nu_0^\Lambda \varepsilon} \left(\frac{1}{D} + \frac{N}{D'} + \frac{1}{\tilde{D}} \right) + \frac{1}{D_2} \right] \|\partial_l^j h^\perp\|_\Lambda^2 \\
&\quad + \frac{D' \nu_1^\Lambda}{2\nu_0^\Lambda \varepsilon} \sum_{i, c_i(j) > 0} \|\partial_{l+\delta_i}^{j-\delta_i} h^\perp\|_\Lambda^2 \\
&\quad + \left[\frac{DC_{\pi k}}{2\varepsilon} + \frac{D'C_{\pi k}}{\varepsilon} + \frac{\tilde{D}C_{\pi k}}{\varepsilon} \right] \sum_{|l'| \leq k-1} \|\partial_{l'}^0 h\|_{L_{x,v}^2}^2 \\
&\quad + \frac{2(C(\delta) + \nu_6^\Lambda)}{\varepsilon^2} \|h^\perp\|_{H_{x,v}^{k-1}}^2 \\
&\quad + \frac{D_2}{\varepsilon} (\mathcal{G}_{x,v}^k(h, h))^2.
\end{aligned}$$

And in the case $|j| = 1$,

$$\begin{aligned}
\frac{d}{dt} \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_{L_{x,v}^2}^2 &\leq \left[\frac{2}{\varepsilon^2} \left(\frac{\nu_1^\Lambda \delta}{\nu_0^\Lambda} - \nu_5^\Lambda \right) + \frac{\nu_1^\Lambda}{\nu_0^\Lambda \varepsilon} \left(\frac{1}{D} + \frac{1}{D'} + \frac{1}{\tilde{D}} \right) + \frac{1}{D_2} \right] \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_\Lambda^2 \\
&\quad + \left[\frac{DC_{\pi k}}{\varepsilon} + \frac{D'}{\varepsilon} + \frac{\tilde{D}C_{\pi k}}{\varepsilon} \right] \sum_{|l'|=k} \|\partial_{l'}^0 h\|_{L_{x,v}^2}^2 \\
&\quad + \frac{2(C(\delta) + \nu_6^\Lambda)}{\varepsilon^2} \|h^\perp\|_{H_{x,v}^{k-1}}^2 \\
&\quad + \frac{D_2}{\varepsilon} (\mathcal{G}_{x,v}^k(h, h))^2.
\end{aligned}$$

By taking $D = \tilde{D} = 9\nu_1^\Lambda \varepsilon / \nu_0^\Lambda \nu_5^\Lambda$, $D_2 = 3\varepsilon / \nu_5^\Lambda$, $\delta = \nu_0^\Lambda \nu_5^\Lambda / 6\nu_1^\Lambda$ and $D' = 9\nu_1^\Lambda \varepsilon / \nu_0^\Lambda \nu_5^\Lambda$, if $|j| = 1$, or $D' = 9\nu_1^\Lambda N \varepsilon / \nu_0^\Lambda \nu_5^\Lambda$, if $|j| > 1$, we obtain (3.16) and (3.17).

B.2.9. A new time evolution of $\langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2}$. By integrating by part in x then in v we obtain the following equality on the evolution of the scalar product.

$$\frac{d}{dt} \langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2} = 2 \langle \partial_{l-\delta_i}^{\delta_i} G_\varepsilon(h), \partial_l^0 h \rangle_{L_{x,v}^2} + \frac{2}{\varepsilon} \langle \partial_{l-\delta_i}^{\delta_i} \Gamma(h, h), \partial_l^0 h \rangle_{L_{x,v}^2}.$$

We will bound above the first term as in the previous case and for the second term involving Γ we use (H4) and Young's inequality with a constant $D_3 > 0$. Moreover, we decompose $\partial_l^0 h$ into its fluid part and its microscopic part and we apply (3.4) on the fluid part. This raises

$$2\langle \partial_{l-\delta_i}^{\delta_i} \Gamma(h, h), \partial_l^0 h \rangle_{L_{x,v}^2} \leq D_3 (\mathcal{G}_{x,v}^k(h, h))^2 + \frac{1}{D_3} \|\partial_l^0 h^\perp\|_\Lambda^2 + \frac{C_\pi}{D_3} \|\partial_l^0 h\|_{L_{x,v}^2}^2.$$

Finally we obtain an upper bound for the time-derivative:

$$\begin{aligned} \frac{d}{dt} \langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2} &\leq \left[\frac{C^L \eta}{\varepsilon^2} + \frac{1}{D_3} \right] \|\partial_l^0 h^\perp\|_\Lambda^2 + \frac{C^L}{\eta \varepsilon^2} \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2 + \left(\frac{C_\pi}{\varepsilon D_3} - \frac{1}{\varepsilon} \right) \|\partial_l^0 h\|_{L_{x,v}^2}^2 \\ &\quad + \frac{D_3}{\varepsilon} (\mathcal{G}_{x,v}^k(h, h))^2. \end{aligned}$$

Now we can use the properties of π_L concerning the v -derivatives, equation (3.3), the equivalence of norm under the projection π_L , equation (3.4), and Poincaré inequality get the following upper bound:

$$\begin{aligned} \|\partial_{l-\delta_i}^{\delta_i} h\|_\Lambda^2 &\leq 2 \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_\Lambda^2 + 2 \|\partial_{l-\delta_i}^{\delta_i} \pi_L(h)\|_\Lambda^2 \\ &\leq 2 \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_\Lambda^2 + 2C_{\pi k} C_\pi \|\partial_{l-\delta_i}^0(h)\|_{L_{x,v}^2}^2 \\ &\leq 2 \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_\Lambda^2 + 2C_{\pi k} C_\pi \sum_{|l'| \leq k-1} \|\partial_{l'}^0 h\|_{L_{x,v}^2}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{dt} \langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h \rangle_{L_{x,v}^2} &\leq \left[\frac{C^L \eta}{\varepsilon^2} + \frac{1}{D_3} \right] \|\partial_l^0 h^\perp\|_\Lambda^2 + \frac{2C^L}{\eta \varepsilon^2} \|\partial_{l-\delta_i}^{\delta_i} h^\perp\|_\Lambda^2 + \left(\frac{C_\pi}{\varepsilon D_3} - \frac{1}{\varepsilon} \right) \|\partial_l^0 h\|_{L_{x,v}^2}^2 \\ &\quad + \frac{2C^L C_{\pi k} C_\pi}{\eta \varepsilon^2} \sum_{|l'| \leq k-1} \|\partial_{l'}^0 h\|_{L_{x,v}^2}^2 + \frac{D_3}{\varepsilon} (\mathcal{G}_{x,v}^k(h, h))^2. \end{aligned}$$

We finally define $\eta = 8eC^L C_{\pi k} C_\pi N / \varepsilon$, with $e > 1$, and $D_3 = 2C_\pi$ to yield equation (3.18).

APPENDIX C. APPENDIX 3: PROOF OF THE HYDRODYNAMICAL LIMIT LEMMAS

In this section we are going to prove all the different lemmas used in section 9. All along the demonstration we will use this inequality:

$$(C.1) \quad \forall t > 0, k \in \mathbb{N}^*, q \geq 0, p > 0, t^q k^{2p} e^{-atk^2} \leq C_p(a) t^{q-p}.$$

C.1. Study of the linear part.

C.1.1. *Proof of Lemma 8.3.* Fix T in $[0, +\infty]$. By integrating we compute

$$\begin{aligned} \int_0^T U_{0j}^\varepsilon h_{in} dt &= \sum_{n \in \mathbb{Z}^N - \{0\}} e^{in \cdot x} \left[\int_0^T e^{\frac{i\alpha_j t |n|}{\varepsilon} - \beta_j t |n|^2} dt \right] P_{0j} \left(\frac{n}{|n|} \right) \hat{h}_{in}(n, v) \\ &= \sum_{n \in \mathbb{Z}^N - \{0\}} e^{in \cdot x} \frac{\varepsilon}{i\alpha_j |n| - \varepsilon \beta_j |n|^2} \left[e^{\frac{i\alpha_j T |n|}{\varepsilon} - \beta_j T |n|^2} - 1 \right] P_{0j} \hat{h}_{in}(n, v). \end{aligned}$$

The Fourier transform is an isometry in L_x^2 and therefore

$$\left\| \int_0^T U_{0j}^\varepsilon h_{in} dt \right\|_{L_x^2 L_v^2}^2 \leq \varepsilon^2 \sum_{n \in \mathbb{Z}^N - \{0\}} \frac{2}{\alpha_j^2 |n|^2 + \varepsilon^2 \beta_j^2 |n|^4} \left\| P_{0j} \left(\frac{n}{|n|} \right) \hat{h}_{in}(n, v) \right\|_{L_v^2}^2.$$

Finally, we know that, like e_{0j} , P_{0j} is continuous on the compact \mathbb{S}^{N-1} and so is bounded. But the latter is a linear operator acting on L_v^2 and therefore it is bounded by M_{0j} in the operator norm on L_v^2 . Thus

$$\begin{aligned} \left\| \int_0^T U_{0j}^\varepsilon h_{in} dt \right\|_{L_x^2 L_v^2}^2 &\leq \varepsilon^2 \frac{M_{0j}^2}{\alpha_j^2} \sum_{n \in \mathbb{Z}^N - \{0\}} \left\| \hat{h}_{in}(n, v) \right\|_{L_v^2}^2 \\ &\leq \varepsilon^2 \frac{M_{0j}^2}{\alpha_j^2} \|h_{in}(x, v)\|_{L_x^2 L_v^2}^2, \end{aligned}$$

which is the expected result.

Now, let us look at the L_x^2 -norm of this operator, to see how the torus case is different from the case \mathbb{R}^N studied in [3] and [9].

Consider a direction n_1 in the Fourier transform space of the torus and define $\phi_{n_1} = \mathcal{F}_x^{-1}(e^{in_1 \cdot})$. We have the following equality

$$\langle U_{0j}^\varepsilon h_{in}, \phi_{n_1} \rangle_{L_x^2} = \langle \hat{U}_{0j}^\varepsilon \hat{h}_{in}, \hat{\phi}_{n_1} \rangle_{L_n^2} = e^{\frac{i\alpha_j t |n_1|}{\varepsilon} - \beta_j t |n_1|^2} P_{0j} \left(\frac{n_1}{|n_1|} \right) \hat{h}_{in}(n_1, v).$$

If we do not integrate in time, one can easily see that this expression cannot have a limit as ε tends to 0 if $P_{0j} \left(\frac{n_1}{|n_1|} \right) \hat{h}_{in}(n_1, v) \neq 0$, and so we cannot even have a weak convergence. Therefore we have a convergence without averaging in time if and only if $P_{0j} \left(\frac{n_1}{|n_1|} \right) \hat{h}_{in}(n_1, v) = 0$, for all $j = \pm 1$ and all direction n_1 . This means that for all $j = \pm 1$ and all n_1 , $\langle e_{0j} \left(\frac{n_1}{|n_1|} \right), \hat{h}_{in} \rangle_{L_v^2} = 0$. By the expression known (see theorem 8.1) of $e_{0\pm 1}$, this is true if and only if $\nabla_x \cdot u_{in} = 0$ and $\rho_{in} + \theta_{in} = 0$.

C.1.2. *Proof of Lemma 8.4.* This lemma deals with three different terms and we study them one by one because their behaviour are quite different.

The term U_{1j}^ε . :

We remind that we have

$$\hat{U}_{1j}^\varepsilon \hat{h}_{in} = \chi_{|\varepsilon n| \leq n_0} e^{\frac{i\alpha_j t |n|}{\varepsilon} - \beta_j t |n|^2} \left(e^{\frac{t}{\varepsilon^2} \gamma_j(|\varepsilon n|)} - 1 \right) P_{0j} \left(\frac{n}{|n|} \right) \hat{h}_{in}(n, v).$$

If we take $T > 0$, by Parseval identity we get

$$\left\| \int_0^T U_{1j}^\varepsilon h_{in} dt \right\|_{L_x^2 L_v^2}^2 = \sum_{n \in \mathbb{Z}^N - \{0\}} \chi_{|\varepsilon n| \leq n_0} \left| \int_0^T e^{\frac{i\alpha_j t |n|}{\varepsilon} - \beta_j t |n|^2} \left(e^{\frac{t}{\varepsilon^2} \gamma_j(|\varepsilon n|)} - 1 \right) dt \right|^2 \left\| P_{0j} \hat{h}_{in} \right\|_{L_v^2}^2.$$

But then we can use the fact that $|e^a - 1| \leq |a| e^{|a|}$, the inequalities satisfied by γ_j and the computational inequality (C.1) to obtain

$$\begin{aligned} \left| \int_0^T e^{\frac{i\alpha_j t |n|}{\varepsilon} - \beta_j t |n|^2} \left(e^{\frac{t}{\varepsilon^2} \gamma_j(|\varepsilon n|)} - 1 \right) dt \right| &\leq C_\gamma \varepsilon \int_0^T t |n|^3 e^{-\frac{t\beta_j}{2} |n|^2} dt \\ &\leq C_\gamma \varepsilon C_{3/2} \left(\frac{\beta_j}{4} \right) \int_0^T \frac{1}{\sqrt{t}} e^{-\frac{t\beta_j}{4} |n|^2} dt \\ &\leq C_\gamma \varepsilon C_{3/2} \left(\frac{\beta_j}{4} \right) \int_0^{+\infty} \frac{1}{\sqrt{t}} e^{-\frac{t\beta_j}{4} |n|^2} dt, \end{aligned}$$

which is independent of n and is written $I\varepsilon$. Therefore we have the expected inequality, by using the continuity of P_{0j} ,

$$\left\| \int_0^T U_{1j}^\varepsilon h_{in} dt \right\|_{L_x^2 L_v^2}^2 \leq \varepsilon^2 I^2 M_{0j}^2 \|h_{in}\|_{L_x^2 L_v^2}^2.$$

The last two inequalities we want to show comes from Parseval's identity, the properties of γ_j and the computational inequality (C.1):

$$\begin{aligned} \left\| U_{1j}^\varepsilon h_{in} \right\|_{L_x^2 L_v^2}^2 &= \sum_{n \in \mathbb{Z}^N - \{0\}} \chi_{|\varepsilon n| \leq n_0} e^{-2\beta_j t |n|^2} \left| e^{\frac{t}{\varepsilon^2} \gamma_j(|\varepsilon n|)} - 1 \right|^2 \left\| P_{0j} \left(\frac{n}{|n|} \right) \hat{h}_{in} \right\|_{L_v^2}^2 \\ &\leq M_{0j}^2 C_\gamma^2 \varepsilon^2 \sum_{n \in \mathbb{Z}^N - \{0\}} \chi_{|\varepsilon n| \leq n_0} t^2 |n|^6 e^{-\beta_j t |n|^2} \left\| \hat{h}_{in} \right\|_{L_v^2}^2 \\ (C.2) \quad &\leq M_{0j}^2 C_\gamma^2 \varepsilon^2 C_2 \left(\frac{\beta_j}{2} \right) \sum_{n \in \mathbb{Z}^N - \{0\}} \chi_{|\varepsilon n| \leq n_0} |n|^2 e^{-\frac{\beta_j t}{2} |n|^2} \left\| \hat{h}_{in} \right\|_{L_v^2}^2. \end{aligned}$$

Finally, if we integrate in t between 0 and $+\infty$ we obtain the expected second inequality of the lemma. If we merely bound $e^{-\frac{\beta_j t}{2} |n|^2}$ by one and use the fact that $\chi_{|\varepsilon n| \leq n_0} \leq 1$ and $\chi_{|\varepsilon n| \leq n_0} \varepsilon^2 |n|^2 \leq n_0^2$ we obtain the third inequality of the lemma for $\delta = 1$ and $\delta = 0$. Then by interpolation we obtain the general case for $0 \leq \delta \leq 1$.

The term U_{2j}^ε :

Fix $T > 0$. By Parseval's identity we have

$$\begin{aligned} \left\| \int_0^T U_{2j}^\varepsilon h_{in} dt \right\|_{L_x^2 L_v^2}^2 &= \sum_{n \in \mathbb{Z}^N - \{0\}} \chi_{|\varepsilon n| \leq n_0} \left| \int_0^T e^{\frac{i\alpha_j t |n|}{\varepsilon} - \beta_j t |n|^2 + \frac{t}{\varepsilon^2} \gamma_j(|\varepsilon n|)} dt \right|^2 |\varepsilon n|^2 \left\| \tilde{P}_{1j} \hat{h}_{in} \right\|_{L_v^2}^2 \\ &\leq \sum_{n \in \mathbb{Z}^N - \{0\}} \frac{4}{\beta_j^2 |n|^4} |\varepsilon n|^2 \left\| \tilde{P}_{1j} \left(|\varepsilon n|, \frac{n}{|n|} \right) \hat{h}_{in} \right\|_{L_v^2}^2, \end{aligned}$$

where we used the inequalities satisfied by γ and integration in time.

Then, \tilde{P}_{1j} is continuous on the compact $[-n_0, n_0] \times \mathbb{S}^{N-1}$ and so is bounded, as an operator acting on L_v^2 , by $M_{1j} > 0$. Hence, Parseval's identity offers us the first inequality of the lemma.

The last two inequalities are just using Parseval's identity and the continuity of \tilde{P}_{1j} . Indeed,

$$\begin{aligned} \|U_{2j}^\varepsilon h_{in}\|_{L_x^2 L_v^2}^2 &= \sum_{n \in \mathbb{Z}^N - \{0\}} \chi_{|\varepsilon n| \leq n_0} \left| e^{\frac{i\alpha_j t |n|}{\varepsilon} - \beta_j t |n|^2 + \frac{t}{\varepsilon^2} \gamma_j(|\varepsilon n|)} \right|^2 |\varepsilon n|^2 \left\| \tilde{P}_{1j}(n) \hat{h}_{in} \right\|_{L_v^2}^2 \\ &\leq M_{1j}^2 \varepsilon^2 \sum_{n \in \mathbb{Z}^N - \{0\}} \chi_{|\varepsilon n| \leq n_0} |n|^2 e^{-t\beta_j |n|^2} \left\| \hat{h}_{in} \right\|_{L_v^2}^2. \end{aligned}$$

We recognize here the same form of inequality (C.2). Thus, we obtain the last two inequalities of the statement in the same way.

The term U_{3j}^ε .

We remind the reader that

$$\hat{U}_{3j}^\varepsilon = (\chi_{|\varepsilon n| \leq n_0} - 1) e^{\frac{i\alpha_j t |n|}{\varepsilon} - \beta_j t |n|^2} P_{0j} \left(\frac{n}{|n|} \right).$$

We have the following inequality

$$|\chi_{|\varepsilon n| \leq n_0} - 1| \leq \frac{\varepsilon n}{n_0}.$$

Therefore, replacing \tilde{P}_{1j} by $\frac{1}{n_0} P_{0j}$ and β_j by $2\beta_j$ (since $\frac{t}{\varepsilon^2} \gamma_j(|\varepsilon n|) \leq \frac{t\beta_j}{2} |n|^2$) in the proof made for U_{2j}^ε , we obtain the expected three inequalities for $U_{3j}^\varepsilon h_{in}$, the last one only with $\delta = 1$.

To have the last inequality in δ , it is enough to bound $|\chi_{|\varepsilon n| \leq n_0} - 1|$ by 1 and then using the continuity of P_{0j} to have the result for $\delta = 0$. Finally, we interpolate to get the general result for all $0 \leq \delta \leq 1$.

C.1.3. *Proof of Lemma 8.5.* Thanks to Theorem 8.1 we have that

$$\|U_R^\varepsilon h_{in}\|_{L_x^2 L_v^2}^2 = \left\| \hat{U}_R(t/\varepsilon^2, \varepsilon n, v) \hat{h}_{in} \right\|_{L_n^2 L_v^2}^2 \leq C_R^2 e^{-2\frac{\sigma t}{\varepsilon^2}} \|h_{in}\|_{L_x^2 L_v^2}^2.$$

But then we have, thanks to the technical lemma C.1, that $e^{-2\frac{\sigma t}{\varepsilon^2}} \leq C_{1/2} (2\sigma)^{\frac{\varepsilon}{\sqrt{t}}}$, which gives us the last two inequalities we wanted. For the first inequality, a mere Cauchy-Schwartz inequality raises

$$\left\| \int_0^T U_R^\varepsilon h_{in} dt \right\|_{L_x^2 L_v^2}^2 \leq T \int_0^T \|U_R^\varepsilon h_{in}\|_{L_x^2 L_v^2}^2 dt,$$

which gives us the first inequality by integrating in t .

Now, let us suppose that we have the strong convergence down to $t_0 = 0$. At $t = 0$ we can write that $e^{tG_\varepsilon} = Id$ and therefore that:

$$Id = \chi_{|\varepsilon n| \leq n_0} \sum_{j=-1}^2 P_j \left(|\varepsilon n|, \frac{n}{|n|} \right) + \hat{U}_R(0, \varepsilon n, v).$$

We have the strong convergence down to 0 as ε tends to 0. Therefore, taking the latter equality at $\varepsilon = 0$ we have, because $\sum_{j=-1}^2 P_{0j} = \pi_L$,

$$\hat{U}_R(0, 0, v) = Id - \pi_L.$$

Then $\hat{U}_R \hat{h}_{in}$ tends to 0 as ε tends to 0 in $C([0, +\infty), L_x^2 L_v^2)$ if and only if h_{in} belongs to $\text{Ker}(L)$.

In that case, we can use the proof of Lemma 6.2 of [3] in which they noticed that

$$U_R^\varepsilon(t, x, v) = e^{tG_\varepsilon} U_R^\varepsilon(0, x, v) = e^{tG_\varepsilon} \left[\mathcal{F}_x^{-1} \left(Id - \chi_{|\varepsilon n| \leq n_0} \sum_{j=-1}^2 P_j(\varepsilon n) \right) \mathcal{F}_x \right].$$

Thanks to that new form we have that, if $h_{in} = \pi_L(h_{in})$,

$$U_R^\varepsilon(t, x, v) h_{in} = e^{tG_\varepsilon} \left[\mathcal{F}_x^{-1} \left((1 - \chi_{|\varepsilon n| \leq n_0}) - |\varepsilon n| \chi_{|\varepsilon n| \leq n_0} \sum_{j=-1}^2 \tilde{P}_{1j}(\varepsilon n) \right) \hat{h}_{in} \right],$$

because $\pi_L = \sum_{j=-1}^2 P_{0j}$.

Therefore we can redo the same estimates we worked out in the previous lemmas and use the same interpolation method to get the result stated in Lemma 8.5.

C.2. Study of the bilinear part.

C.2.1. A simplification without loss of generality. All the terms, apart from the remainder term, we are about to study are of the following form

$$\psi_{ij}^\varepsilon(u_\varepsilon) = \int_0^t \sum_{n \in \mathbb{Z}^N - \{0\}} g(t, s, k, x) P(n) \hat{u}_\varepsilon(s, k, v) ds,$$

with $P(n)$ being a projector in L_v^2 , bounded uniformly in n .

Looking at the dual definition of the norm of a function in $L_{x,v}^2$, we can consider f in $C_c^\infty(\mathbb{T}^N \times \mathbb{R}^N)$ such that $\|f\|_{L_{x,v}^2} = 1$ and take the scalar product with $\psi_{ij}^\varepsilon(u_\varepsilon)$. This yields, since P is a projector and thus symmetric,

$$\begin{aligned} \langle \psi_{ij}^\varepsilon(u_\varepsilon), f \rangle_{L_{x,v}^2} &= \int_{\mathbb{T}^N} \int_0^t \sum_{n \in \mathbb{Z}^N - \{0\}} g(t, s, k, x) \langle P(n) \hat{u}_\varepsilon, f \rangle_{L_v^2} ds \\ (C.3) \quad &= \int_{\mathbb{T}^N} \int_0^t \sum_{n \in \mathbb{Z}^N - \{0\}} g(t, s, k, x) \langle \hat{u}_\varepsilon, P(n) f \rangle_{L_v^2} ds. \end{aligned}$$

But then, using the computations made in subsection A.2.2 one can easily see that

$$(C.4) \quad \langle \hat{u}_\varepsilon, P(n) f \rangle_{L_v^2} \leq \|h\|_{L_v^2}^2 \|(1 + |v|)^\gamma P(n) f\|_{L_v^2}.$$

Finally, we know that L is of the form $K - \Lambda(v)$ (see [20] for instance) with $\Lambda(v) \sim (1 + |v|)^\gamma$ in the case of hard potentials (see [10] for instance). Therefore, we have the following bound

$$(1 + |v|)^{2\gamma} e^{(t-s)G_\varepsilon} \leq C e^{(t-s)\left(\frac{1}{\varepsilon^2}(K - \Lambda/2) - \frac{1}{\varepsilon}v \cdot \nabla_x\right)}.$$

In terms of Fourier coefficients in x , this implies that $(1 + |v|)^\gamma P(n)$ is still uniformly bounded in n as an operator in L_v^2 , as well as for $(1 + |v|)^\gamma U_R^\varepsilon$. Thus, combining (C.3), (C.4) and the last remark shows us that without loss of generality, we can consider that the following holds (even for the remainder term):

$$\psi_{ij}^\varepsilon(u_\varepsilon) = \int_0^t \sum_{n \in \mathbb{Z}^N - \{0\}} g(t, s, k, x) \hat{f}_\varepsilon(s, k, v) ds,$$

with

$$\|\hat{f}_\varepsilon\|_{L_v^2} \leq M_{ij} \|\hat{h}_\varepsilon\|_{L_v^2}^2.$$

C.2.2. Proof of Lemma 8.6. For the first inequality, fix $T > 0$ and integrate by part in t to obtain

$$\begin{aligned} \int_0^T \psi_{0j}^\varepsilon(u_\varepsilon) dt &= \sum_{n \in \mathbb{Z}^N - \{0\}} e^{in \cdot x} \int_0^T \left(\int_0^t e^{-i \frac{\alpha_j(t-s)}{\varepsilon} |n| - (t-s)\beta_j |n|^2} |n| \hat{f}_\varepsilon(s) ds \right) dt \\ &= \sum_{n \in \mathbb{Z}^N - \{0\}} e^{in \cdot x} \frac{\varepsilon}{i\alpha_j |n| - \varepsilon\beta_j |n|} \left[\int_0^T \left(e^{i \frac{\alpha_j(T-s)}{\varepsilon} |n| - (T-s)\beta_j |n|^2} - 1 \right) \hat{f}_\varepsilon(s) ds \right]. \end{aligned}$$

Finally we can use Parseval's identity and a Cauchy-Schwarz to get

$$\begin{aligned} \left\| \int_0^T \psi_{0j}^\varepsilon(u_\varepsilon) dt \right\|_{L_x^2 L_v^2}^2 &\leq \sum_{n \in \mathbb{Z}^N - \{0\}} \frac{\varepsilon^2}{\varepsilon^2 \beta_j^2 |n|^2 + \alpha_j^2} T \int_0^T 2 \left\| \hat{f}_\varepsilon(s, n, v) \right\|_{L_v^2}^2 ds \\ &\leq \frac{2M_{1j}^2}{\alpha_j^2} T^2 \varepsilon^2 \sup_{t \in [0, T]} \|h_\varepsilon(t, x, v)\|_{L_x^2 L_v^2}^4, \end{aligned}$$

where we used the subsection above and Parseval's identity again. This is exactly the expected result.

C.2.3. Proof of Lemma 8.8. We divide this proof in three paragraphs, each of them studying a different term.

The term ψ_{1j}^ε . :

We will just prove the last inequality and then merely applying Cauchy-Schwarz inequality will lead to the first two.

Fix $T > 0$. By a change of variable we can write

$$\psi_{1j}^\varepsilon(u_\varepsilon) = \sum_{n \in \mathbb{Z}^N - \{0\}} e^{ik \cdot x} \chi_{|\varepsilon n| \leq n_0} \int_0^T e^{\frac{i\alpha_j s}{\varepsilon} |n| - \beta_j s |n|^2} \left(e^{\frac{s}{\varepsilon^2} \gamma_j(|\varepsilon n|)} - 1 \right) |n| \hat{f}_\varepsilon(T - s) ds.$$

By the study made in the proof of Lemma 8.4 we have that

$$\begin{aligned} & \left| \int_0^T e^{\frac{i\alpha_j s}{\varepsilon}|n|-\beta_j s|n|^2} \left(e^{\frac{s}{\varepsilon^2}\gamma_j(|\varepsilon n|)} - 1 \right) |n| \hat{f}_\varepsilon(T-s) ds \right| \\ & \leq C_\gamma |n|^4 \varepsilon \int_0^T s e^{-\frac{\beta_j s}{2}|n|^2} \left| \hat{f}_\varepsilon(T-s) \right| ds. \end{aligned}$$

Then we use the computational inequality (C.1) and a Cauchy-Schwarz to raise

$$\begin{aligned} & \left| \int_0^T e^{\frac{i\alpha_j s}{\varepsilon}|n|-\beta_j s|n|^2} \left(e^{\frac{s}{\varepsilon^2}\gamma_j(|\varepsilon n|)} - 1 \right) |n| \hat{u}_\varepsilon ds \right| \\ & \leq \varepsilon C_\gamma C_1 \left(\frac{\beta_j}{4} \right) |n|^2 \int_0^T e^{-\frac{\beta_j s}{4}|n|^2} \left| \hat{f}_\varepsilon \right| ds \\ & \leq \varepsilon C_\gamma C_1 \left(\frac{\beta_j}{4} \right) |n|^2 \sqrt{\frac{4}{\beta_j |n|^2}} \left[\int_0^T e^{-\frac{\beta_j s}{4}|n|^2} \left| \hat{f}_\varepsilon \right|^2 ds \right]^{1/2}. \end{aligned} \quad (\text{C.5})$$

We can finish by using Parseval's identity, denoting C a constant independant of ε and T , the continuity of P_{1j} and the computational inequality (C.1).

$$\begin{aligned} \|\psi_{1j}^\varepsilon(u_\varepsilon)(T)\|_{L_x^2 L_v^2}^2 & \leq C \sum_{n \in \mathbb{Z}^N - \{0\}} \chi_{|\varepsilon n| \leq n_0} \varepsilon^2 |n|^2 \int_0^T e^{-\frac{\beta_j s}{4}|n|^2} \left\| \hat{f}_\varepsilon(T-s) \right\|_{L_v^2}^2 ds \\ & \leq C k_0 M_{1j}^2 \varepsilon \sum_{n \in \mathbb{Z}^N - \{0\}} \int_0^T |n| e^{-\frac{\beta_j s}{2}|n|^2} \left\| \hat{h}_\varepsilon(T-s) \right\|_{L_v^2}^4 ds \\ & \leq C k_0 M_{1j}^2 \varepsilon \int_0^T \frac{1}{\sqrt{s}} \|h_\varepsilon(T-s)\|_{L_x^2 L_v^2}^4 ds, \end{aligned}$$

which gives us the expected inequality by considering the supremum of the norm of h_ε .

The term ψ_{2j}^ε :

As in the case ψ_{1j}^ε , we are going to prove the third inequality only.

Fix $T > 0$, a change of variable gives us

$$\psi_{2j}^\varepsilon(u_\varepsilon) = \sum_{n \in \mathbb{Z}^N - \{0\}} e^{ik \cdot x} \chi_{|\varepsilon n| \leq n_0} \int_0^T e^{\frac{i\alpha_j s}{\varepsilon}|n|-\beta_j s|n|^2 + \frac{s}{\varepsilon^2}\gamma_j(|\varepsilon n|)} \varepsilon |n|^2 \hat{f}_\varepsilon(T-s) ds.$$

We can see that

$$\left| \int_0^T e^{\frac{i\alpha_j s}{\varepsilon}|n|-\beta_j s|n|^2 + \frac{s}{\varepsilon^2}\gamma_j(|\varepsilon n|)} \varepsilon |n|^2 \hat{f}_\varepsilon(T-s) ds \right| \leq \varepsilon |n|^2 \int_0^T e^{-\frac{\beta_j s}{2}|n|^2} \left| \hat{f}_\varepsilon(T-s) \right| ds.$$

This bound is of the same form as equation (C.5). Therefore we have the same result.

The term ψ_{3j}^ε :

As above, we will show the third inequality only.

Fix $T > 0$, we can write

$$\psi_{3j}^\varepsilon(u_\varepsilon) = \sum_{n \in \mathbb{Z}^N - \{0\}} e^{ik \cdot x} (\chi_{|\varepsilon n| \leq n_0} - 1) \int_0^T e^{\frac{i\alpha_j s}{\varepsilon} |n| - \beta_j s |n|^2} |n| \hat{f}_\varepsilon(T - s, n, v) ds.$$

Looking at the fact that $|\chi_{|\varepsilon n| \leq n_0} - 1| \leq \frac{\varepsilon |n|}{n_0}$, we find the same kind of inequality as equation (C.5). Thus, we reach the same result.

C.2.4. *Proof of Lemma 8.9.* We remind the reader that

$$\Psi_R^\varepsilon(u_\varepsilon) = \int_0^t \frac{1}{\varepsilon} U_R^\varepsilon(t - s) f_\varepsilon(s) ds,$$

and that, by Theorem 8.1,

$$\|U_R^\varepsilon f_\varepsilon\|_{L_x^2 L_v^2}^2 \leq C_R^2 e^{-2\frac{\sigma t}{\varepsilon^2}} \|f_\varepsilon\|_{L_x^2 L_v^2}^2.$$

Hence, a Cauchy-Schwarz inequality gives us the third inequality for $\|\psi_R^\varepsilon(u_\varepsilon)(T)\|_{L_x^2 L_v^2}^2$, and then the two others inequality stated above.

C.2.5. *Proof of Lemma 8.11.* We remind the reader that

$$\Psi(u) = \mathcal{F}_x^{-1} [\psi_{00}^\varepsilon(u) + \psi_{02}^\varepsilon(u)] \mathcal{F}_x.$$

As above, and because in that case $\alpha_j = 0$, we can write $\psi_{0j}^\varepsilon(u_\varepsilon - u)(T)$, for some $T > 0$, and apply a Cauchy-Schwarz inequality:

$$\begin{aligned} \|\psi_{0j}^\varepsilon(u_\varepsilon - u)\|_{L_x^2 L_v^2}^2(T) &= \sum_{n \in \mathbb{Z}^N - \{0\}} |n|^2 \int_{\mathbb{R}^N} \left| \int_0^T e^{-s\beta_j |n|^2} P_{1j} \hat{\Gamma}(h_\varepsilon - h, h_\varepsilon + h) ds \right|^2 dv \\ &\leq \frac{M_{1j}^2}{\beta_j^2} \sup_{t \in [0, T]} \|\Gamma(h_\varepsilon - h, h_\varepsilon + h)\|_{L_x^2 L_v^2}^2. \end{aligned}$$

But because \mathbb{T}^N is bounded in \mathbb{R}^N and thanks to (H4) and the boundedness of $(h_\varepsilon)_\varepsilon$ and h (both bounded by M) in $H_x^k L_v^2$ (Theorem 2.3), we can have the following control:

$$\|\Gamma(h_\varepsilon - h, h_\varepsilon + h)\|_{L_x^2 L_v^2}^2 \leq 4M^2 C_\Gamma^2 \text{Volume}(\mathbb{T}^N) \|h_\varepsilon - h\|_{L_x^\infty L_v^2}.$$

Therefore we obtain the last inequality and the first two just come from Cauchy-Schwarz inequality.

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